Appendices

APPENDIX A: DETAILED STATEMENT OF MODEL 1

Production

With fixed production factors and full employment, output is fixed in each region *z*: $XS_z = \overline{XS}_z$, where XS_z is domestic production in region *z*. That constraint, however, is not part of the model; rather, it is a closure equation.

Income and savings

Regional income is equal to the value of production.

$$Y_z = P_z \ XS_z \tag{30}$$

where

P_z Producer price in region z

Y_z Income in region z

There is no distinction between consumption and investment. It follows that savings, the difference between income and consumption, are equal to the current account balance (CAB). So the regional agent's budget constraint is

$$CAB_z = Y_z - PC_z \, Q_z \tag{31}$$

where

CAB_z Current account balance of region *z*

PC_z Price of the composite good in region *z*

Q_z Domestic demand for the composite good in region *z*

To make the model more compact, we substitute (30) into (31), which becomes

$$CAB_z = P_z \ XS_z - PC_z \, Q_z \tag{3}$$

We can eliminate equation (30) and variable Y_z from the model.

Not only is the current account balance equal to savings, but it is by definition equal to the difference between the aggregate value of exports and the aggregate value of imports.

$$CAB_z = PXT_z \ EXT_z - PMT_z \ IMT_z \tag{4}$$

where

IMT_z Total imports of region z

EXT_z Total exports of region z

PMT_z Price of composite imports in region z

PXT_z Price of composite exports of region z

Trade

Production is allocated between sales on the domestic market and exports so as to maximize its value subject to a CET transformation function.

$$XS_z = B_z\left[\beta_z D_z^{\kappa_z} + (1-\beta_z) EXT_z^{\kappa_z}\right]^{\frac{1}{\kappa_z}} \text{ where } \kappa_z = \frac{\tau_z + 1}{\tau_z}, \text{ with } 0 < \tau_z < \infty \tag{5}$$

$$\frac{EXT_z}{D_z} = \left(\frac{\beta_z}{1-\beta_z}\frac{PXT_z}{PL_z}\right)^{\tau_z} \tag{6}$$

where

D_z Domestic demand for the locally produced good in region z

PL_z Market price of local product in region z

and

B_z Scale parameter, CET product aggregator

β_z Share parameter, CET product aggregator

τ_z Elasticity of transformation: $0 < \tau_z < \infty$

$\kappa_z = \frac{\tau_z + 1}{\tau_z}$: $1 < \kappa_z < \infty$

Total exports are allocated among destination regions so as to maximize their value subject to a CET transformation function.

$$EXT_z = B_z^X\left[\sum_{zj}\beta_{z,zj}^X\left(EX_{z,zj}\right)^{\kappa_z^X}\right]^{\frac{1}{\kappa_z^X}} \text{ where } \kappa_z^X = \frac{\tau_z^X + 1}{\tau_z^X}, \text{ with } 0 < \tau_z^X < \infty \tag{7}$$

$$EX_{z,zj} = \frac{EXT_z}{\left(B_z^X\right)^{1+\tau_z^X}}\left[\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X\, PXT_z}\right]^{\tau_z^X} \tag{8}$$

where

B_z^X Scale parameter, CET exports aggregator

$\beta_{z,zj}^X$ Share parameter, CET exports aggregator

τ_z^X Elasticity of transformation: $0 < \tau_z^X < \infty$

$\kappa_z^X = \dfrac{\tau_z^X + 1}{\tau_z^X}$: $1 < \kappa_z^X < \infty$

e_z Exchange rate (price of the international currency in terms of region *z*'s currency)

$PW_{z,zj}$ World price of exports from region *z* to region *zj*

$EX_{z,zj}$ Exports by region *z* to region *zj*

Under the Armington hypothesis, domestic demand is distributed between the domestically produced good and imports so as to maximize the quantity acquired, subject to a CES aggregator function.

$$Q_z = A_z\left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}} \text{ where } \rho_z = \frac{1-\sigma_z}{\sigma_z}\text{, with } 0 < \sigma_z < \infty \tag{9}$$

$$\frac{IMT_z}{D_z} = \left(\frac{1-\alpha_z}{\alpha_z}\,\frac{PL_z}{PMT_z}\right)^{\sigma_z} \tag{10}$$

where

A_z Scale parameter, Armington CES function between local production and imports

α_z Share parameter, Armington CES function between local production and imports

σ_z Elasticity of substitution between local production and imports: $0 < \sigma_z < \infty$

$\rho_z = \dfrac{1-\sigma_z}{\sigma_z}$: $-1 < \rho_z < \infty$

Under the Armington hypothesis, imports are distributed among exporting regions so as to maximize the quantity acquired, subject to a CES aggregator function.

$$IMT_z = A_z^M\left[\sum_{zj}\alpha_{zj,z}^M\, IM_{zj,z}^{-\rho_z^M}\right]^{\frac{-1}{\rho_z^M}} \text{ where } \rho_z^M = \frac{1-\sigma_z^M}{\sigma_z^M}\text{, with } 0 < \sigma_z^M < \infty \tag{11}$$

$$IM_{zj,z} = \frac{IMT_z}{\left(A_z^M\right)^{1-\sigma_z^M}}\left[\frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}}\right]^{\sigma_z^M} \tag{12}$$

where

A_z^M Scale parameter, Armington CES function between imports from different regions

$\alpha_{zj,z}^M$ Share parameter, Armington CES function between imports from different regions

σ_z^M Elasticity of substitution between imports from different regions: $0 < \sigma_z^M < \infty$

$\rho_z^M = \dfrac{1 - \sigma_z^M}{\sigma_z^M} : -1 < \rho_z^M < \infty$

Prices

The value of production is equal to the sum of the value of sales on the domestic market and exports.

$$P_z \, XS_z = PL_z D_z + PXT_z \, EXT_z \tag{13}$$

The total value of exports is equal to the sum of values of exports to all regions

$$PXT_z \, EXT_z = e_z \sum_{zj} PW_{z,zj} EX_{z,zj} \tag{14}$$

Total expenditures are equal to the sum of purchases on the domestic market and the value of imports.

$$PC_z Q_z = PL_z D_z + PMT_z \, IMT_z \tag{15}$$

The total value of imorts is equal to the sum of values of imports from all regions

$$PMT_z \, IMT_z = e_z \sum_{zj} PW_{zj,z} IM_{zj,z} \tag{16}$$

Equilibrium

Imports from region *zj* by region *z* must be equal to exports from region *zj* to region *z*.

$$EX_{zj,z} = IM_{zj,z} \tag{17}$$

The world sum of current account balances, expressed in the international currency, must be zero.

$$\sum_z \frac{CAB_z}{e_z} = 0 \tag{18}$$

APPENDIX B: REDUNDANT EQUATIONS IN MODEL 1

We now show that

- equations (7) and (8) together imply (14), which is therefore redundant;
- equations (11) and (12) together imply (16), which is therefore redundant;
- equations (4) and (17) together imply (18), which is therefore redundant;
- equations (13), (15) and (3) together imply (4), which is therefore redundant;
- equations
  - (14) (or equivalently the combination of (7) and (8)),
  - (16) (or equivalently the combination of (11) and (12)),
  - (13), (15) and (3),
  - (18) (or alternatively the combination of (13), (15) and (3), which together imply (4), and equations (4) and (17))

  together imply that, if equation (3) is satisfied for $N-1$ regions, then it is also satisfied for the N^{th} one (Walras' Law). Therefore, one equation of the set (3) may be discarded as redundant.

B.1 Redundancy of equations (14) and (16)

Given (7) and (8), equation (14) is redundant, and given (11) and (12), equation (16) is redundant. This is demonstrated in parallel in three steps.

Step 1: Substitute (8) into (7), and (12) into (11), and develop.

$$EXT_z = B_z^X \left[\sum_{zj} \beta_{z,zj}^X \left(\frac{EXT_z}{\left(B_z^X\right)^{1+\tau_z^X}} \left[\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X \; PXT_z} \right]^{\tau_z^X} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}}$$

$$EXT_z = B_z^X \left[\sum_{zj} \beta_{z,zj}^X \left(\frac{EXT_z}{\left(B_z^X\right)^{1+\tau_z^X}} \left[\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X \; PXT_z} \right]^{\tau_z^X} \right)^{\frac{\tau_z^X+1}{\tau_z^X}} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$EXT_z = B_z^X \left[\left(\frac{EXT_z}{\left(B_z^X\right)^{1+\tau_z^X}} \right)^{\frac{\tau_z^X+1}{\tau_z^X}} \sum_{zj} \beta_{z,zj}^X \left(\left[\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X \; PXT_z} \right]^{\tau_z^X} \right)^{\frac{\tau_z^X+1}{\tau_z^X}} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$EXT_z = B_z^X \left(\frac{EXT_z}{\left(B_z^X\right)^{1+\tau_z^X}} \right) \left[\sum_{zj} \beta_{z,zj}^X \left(\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X \; PXT_z} \right)^{\tau_z^X+1} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$1 = \frac{1}{\left(B_z^X\right)^{\tau_z^X}} \left[\sum_{zj} \beta_{z,zj}^X \left(\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X \; PXT_z} \right)^{\tau_z^X+1} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$IMT_z = A_z^M \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{IMT_z}{\left(A_z^M\right)^{1-\sigma_z^M}} \left[\frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right]^{\sigma_z^M} \right)^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}}$$

$$IMT_z = A_z^M \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{IMT_z}{\left(A_z^M\right)^{1-\sigma_z^M}} \left[\frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right]^{\sigma_z^M} \right)^{-\frac{1-\sigma_z^M}{\sigma_z^M}} \right]^{-\frac{\sigma_z^M}{1-\sigma_z^M}}$$

$$IMT_z = A_z^M \left[\left(\frac{IMT_z}{\left(A_z^M\right)^{1-\sigma_z^M}} \right)^{-\frac{1-\sigma_z^M}{\sigma_z^M}} \sum_{zj} \alpha_{zj,z}^M \left(\left[\frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right]^{\sigma_z^M} \right)^{-\frac{1-\sigma_z^M}{\sigma_z^M}} \right]^{-\frac{\sigma_z^M}{1-\sigma_z^M}}$$

$$IMT_z = A_z^M \left(\frac{IMT_z}{\left(A_z^M\right)^{1-\sigma_z^M}} \right) \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}} \right)^{\sigma_z^M-1} \right]^{-\frac{\sigma_z^M}{1-\sigma_z^M}}$$

$$1 = \frac{1}{\left(A_z^M\right)^{-\sigma_z^M}} \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{e_z PW_{zj,z}}{\alpha_{zj,z}^M PMT_z} \right)^{1-\sigma_z^M} \right]^{-\frac{\sigma_z^M}{1-\sigma_z^M}}$$

$$1 = \frac{1}{\left(B_z^X\right)^{\tau_z^X}} \left[\left(\frac{1}{PXT_z}\right)^{\tau_z^X+1} \sum_{zj} \beta_{z,zj}^X \left(\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X}\right)^{\tau_z^X+1} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$1 = \frac{1}{\left(B_z^X\right)^{\tau_z^X}} \left(\frac{1}{PXT_z}\right)^{\tau_z^X} \left[\sum_{zj} \beta_{z,zj}^X \left(\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X}\right)^{\tau_z^X+1} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$\left(PXT_z\right)^{\tau_z^X} = \frac{1}{\left(B_z^X\right)^{\tau_z^X}} \left[\sum_{zj} \beta_{z,zj}^X \left(\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X}\right)^{\tau_z^X+1} \right]^{\frac{\tau_z^X}{\tau_z^X+1}}$$

$$PXT_z = \frac{1}{B_z^X} \left[\sum_{zj} \beta_{z,zj}^X \left(\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X}\right)^{\tau_z^X+1} \right]^{\frac{1}{\tau_z^X+1}}$$

$$PXT_z = \frac{1}{B_z^X} \left[\sum_{zj} \left(\frac{1}{\beta_{z,zj}^X}\right)^{\tau_z^X} \left(e_z PW_{z,zj}\right)^{\tau_z^X+1} \right]^{\frac{1}{\tau_z^X+1}}$$

$$1 = \frac{1}{\left(A_z^M\right)^{-\sigma_z^M}} \left[\left(\frac{1}{PMT_z}\right)^{1-\sigma_z^M} \sum_{zj} \alpha_{zj,z}^M \left(\frac{e_z PW_{zj,z}}{\alpha_{zj,z}^M}\right)^{1-\sigma_z^M} \right]^{-\frac{\sigma_z^M}{1-\sigma_z^M}}$$

$$1 = \frac{1}{\left(A_z^M\right)^{-\sigma_z^M}} \left(\frac{1}{PMT_z}\right)^{\sigma_z^M} \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{e_z PW_{zj,z}}{\alpha_{zj,z}^M}\right)^{1-\sigma_z^M} \right]^{-\frac{\sigma_z^M}{1-\sigma_z^M}}$$

$$\left(PMT_z\right)^{-\sigma_z^M} = \frac{1}{\left(A_z^M\right)^{-\sigma_z^M}} \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{e_z PW_{zj,z}}{\alpha_{zj,z}^M}\right)^{1-\sigma_z^M} \right]^{-\frac{\sigma_z^M}{1-\sigma_z^M}}$$

$$PMT_z = A_z^M \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{e_z PW_{zj,z}}{\alpha_{zj,z}^M}\right)^{1-\sigma_z^M} \right]^{\frac{1}{1-\sigma_z^M}}$$

$$PMT_z = A_z^M \left[\sum_{zj} \left(\alpha_{zj,z}^M\right)^{\sigma_z^M} \left(e_z PW_{zj,z}\right)^{1-\sigma_z^M} \right]^{\frac{1}{1-\sigma_z^M}}$$

Step 2: Multiply both sides of (8) by $e_z PW_{z,zj}$, and both sides of (12) by $e_z PW_{zj,z}$, and sum over *zj*.

$$e_z \sum_{zj} PW_{z,zj} EX_{z,zj} = e_z \sum_{zj} PW_{z,zj} \frac{EXT_z}{\left(B_z^X\right)^{1+\tau_z^X}} \left[\frac{e_z PW_{z,zj}}{\beta_{z,zj}^X \; PXT_z}\right]^{\tau_z^X}$$

$$e_z \sum_{zj} PW_{z,zj} EX_{z,zj} = \frac{EXT_z}{\left(B_z^X\right)^{1+\tau_z^X}} \left(\frac{1}{PXT_z}\right)^{\tau_z^X} \sum_{zj} \left(e_z PW_{z,zj}\right)^{1+\tau_z^X} \left(\frac{1}{\beta_{z,zj}^X}\right)^{\tau_z^X}$$

$$\sum_{zj} \left(e_z PW_{z,zj}\right)^{1+\tau_z^X} \left(\frac{1}{\beta_{z,zj}^X}\right)^{\tau_z^X} = e_z \sum_{zj} PW_{z,zj} EX_{z,zj} \left(\frac{EXT_z}{\left(B_z^X\right)^{1+\tau_z^X}} \left(\frac{1}{PXT_z}\right)^{\tau_z^X} \right)^{-1}$$

$$e_z \sum_{zj} PW_{zj,z} IM_{zj,z} = \sum_{zj} e_z PW_{zj,z} \frac{IMT_z}{\left(A_z^M\right)^{1-\sigma_z^M}} \left[\frac{\alpha_{zj,z}^M PMT_z}{e_z PW_{zj,z}}\right]^{\sigma_z^M}$$

$$e_z \sum_{zj} PW_{zj,z} IM_{zj,z} = \frac{IMT_z}{\left(A_z^M\right)^{1-\sigma_z^M}} \left(PMT_z\right)^{\sigma_z^M} \sum_{zj} \left(e_z PW_{zj,z}\right)^{1-\sigma_z^M} \left(\alpha_{zj,z}^M\right)^{\sigma_z^M}$$

$$\sum_{zj} \left(e_z PW_{zj,z}\right)^{1-\sigma_z^M} \left(\alpha_{zj,z}^M\right)^{\sigma_z^M} = e_z \sum_{zj} PW_{zj,z} IM_{zj,z} \left(\frac{IMT_z}{\left(A_z^M\right)^{1-\sigma_z^M}} \left(PMT_z\right)^{\sigma_z^M} \right)^{-1}$$

Step 3: Substitute the right-hand side of the last equation in Step 2 into the last equation in Step 1

$$PXT_z = \frac{1}{B_z^X}\left[e_z\sum_{zj} PW_{z,zj}EX_{z,zj}\left(\frac{EXT_z}{(B_z^X)^{1+\tau_z^X}}\left(\frac{1}{PXT_z}\right)^{\tau_z^X}\right)^{-1}\right]^{\frac{1}{\tau_z^X+1}}$$

$$PXT_z = \frac{1}{B_z^X}\left(\frac{EXT_z}{(B_z^X)^{1+\tau_z^X}}\left(\frac{1}{PXT_z}\right)^{\tau_z^X}\right)^{-\frac{1}{\tau_z^X+1}}\left[e_z\sum_{zj} PW_{z,zj}EX_{z,zj}\right]^{\frac{1}{\tau_z^X+1}}$$

$$PXT_z = \frac{1}{B_z^X}\left(\frac{EXT_z}{(B_z^X)^{1+\tau_z^X}}\right)^{-\frac{1}{\tau_z^X+1}}\left(\frac{1}{PXT_z}\right)^{-\frac{\tau_z^X}{\tau_z^X+1}}\left[e_z\sum_{zj} PW_{z,zj}EX_{z,zj}\right]^{\frac{1}{\tau_z^X+1}}$$

$$(PXT_z EXT_z)^{\frac{1}{\tau_z^X+1}} = \left[e_z\sum_{zj} PW_{z,zj}EX_{z,zj}\right]^{\frac{1}{\tau_z^X+1}}$$

$$PXT_z\ EXT_z = e_z\sum_{zj} PW_{z,zj}EX_{z,zj}$$

$$PMT_z = A_z^M\left[e_z\sum_{zj} PW_{zj,z}IM_{zj,z}\left(\frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}}(PMT_z)^{\sigma_z^M}\right)^{-1}\right]^{\frac{1}{1-\sigma_z^M}}$$

$$PMT_z = A_z^M\left(\frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}}(PMT_z)^{\sigma_z^M}\right)^{-\frac{1}{1-\sigma_z^M}}\left[e_z\sum_{zj} PW_{zj,z}IM_{zj,z}\right]^{\frac{1}{1-\sigma_z^M}}$$

$$PMT_z = A_z^M\left(\frac{IMT_z}{(A_z^M)^{1-\sigma_z^M}}\right)^{-\frac{1}{1-\sigma_z^M}}\left(\frac{1}{PMT_z}\right)^{\frac{\sigma_z^M}{1-\sigma_z^M}}\left[e_z\sum_{zj} PW_{zj,z}IM_{zj,z}\right]^{\frac{1}{1-\sigma_z^M}}$$

$$(PMT_z IMT_z)^{\frac{1}{1-\sigma_z^M}} = \left[e_z\sum_{zj} PW_{zj,z}IM_{zj,z}\right]^{\frac{1}{1-\sigma_z^M}}$$

$$PMT_z\ IMT_z = e_z\sum_{zj} PW_{zj,z}IM_{zj,z}$$

The final equations in Step 3 are identical to (14) and (16), which are therefore redundant.

B.2 Equation (18) redundant

Divide (4) throughout by e_z:

$$\frac{CAB_z}{e_z} = \sum_{zj} PW_{z,zj} EX_{z,zj} - \sum_{zj} PW_{zj,z} IM_{zj,z}$$

Sum over z.

$$\sum_z \frac{CAB_z}{e_z} = \sum_z \sum_{zj} PW_{z,zj} EX_{z,zj} - \sum_z \sum_{zj} PW_{zj,z} IM_{zj,z}$$

$$\sum_z \frac{CAB_z}{e_z} = \sum_z \sum_{zj} PW_{z,zj} EX_{z,zj} - \sum_z \sum_{zj} PW_{z,zj} IM_{z,zj}$$

$$\sum_z \frac{CAB_z}{e_z} = \sum_z \sum_{zj} PW_{z,zj} \left(EX_{z,zj} - IM_{z,zj} \right)$$

And, given (17), we have (18). Equations (4) and (17) together imply (18), which is therefore redundant.

B.3 Equation (4) redundant

Substitute (13) and (15) into (3) to obtain (4). Equations (13), (15) and (3) together imply (4), which is therefore redundant.

B.4 Walras' Law

We shall now examine how Walras' Law applies to our model. Define excess demands on the domestic and international markets respectively as

$$XD_z = D_z^D - D_z^O$$

$$XM_{zj,z} = IM_{zj,z} - EX_{zj,z}$$

where

$$D_z^D = (A_z)^{\sigma_z - 1} \left(\frac{\alpha_z PC_z}{PL_z} \right)^{\sigma_z} Q_z \tag{32}$$

$$D_z^O = \left(\frac{1}{B_z} \right)^{\tau_z + 1} \left(\frac{PL_z}{\beta_z P_z} \right)^{\tau_z} XS_z \tag{33}$$

$$IM_{zjj,z} = \left(\frac{1}{A_z^M} \right)^{1-\sigma_z^M} \left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M PMT_z} \right)^{-\sigma_z^M} (A_z)^{\sigma_z - 1} \left(\frac{(1-\alpha_z) PC_z}{PMT_z} \right)^{\sigma_z} Q_z \tag{34}$$

$$EX_{z,zjj} = \left(\frac{1}{B_z^X}\right)^{\tau_z^X+1} \left(\frac{PW_{z,zjj}}{\beta_{z,zjj}^X PXT_z}\right)^{\tau_z^X} \left(\frac{1}{B_z}\right)^{\tau_z+1} \left(\frac{PXT_z}{(1-\beta_z)P_z}\right)^{\tau_z} XS_z \tag{35}$$

Supply and demand equations (32)-(35) are homogeneous with respect to prices. Their mathematical derivations are given in Appendices D-G (look for equations (48), (58), (61) and (64)).

The value of excess demand on the domestic and international markets is given by

$$PL_z XD_z = PL_z D_z^D - PL_z D_z^O$$
$$PW_{zj,z} XM_{zj,z} = PW_{zj,z} IM_{zj,z} - PW_{zj,z} EX_{zj,z}$$

The aggregate value of all excess demands, expressed in terms of the international currency, is

$$\sum_z \frac{PL_z}{e_z} XD_z + \sum_z \sum_{zj} PW_{z,zj}\ XM_{z,zj} =$$
$$\sum_z \frac{PL_z}{e_z} D_z^D - \sum_z \frac{PL_z}{e_z} D_z^O + \sum_z \sum_{zj} PW_{z,zj}\ IM_{z,zj} - \sum_z \sum_{zj} PW_{z,zj}\ EX_{z,zj}$$

Substitute equations (14) and (16) into the last equation. It is true that equations (14) and (16) are no longer present among the model equations (they have already been discarded as redundant), but they are implied[1]:

- equations (7) and (8) together imply (14) (see above and Appendix B);
- equations (11) and (12) together imply (16) (see above and Appendix B).

The substitution yields

$$\sum_z \frac{PL_z}{e_z} XD_z + \sum_z \sum_{zj} PW_{z,zj}\ XM_{z,zj} =$$
$$\sum_z \frac{PL_z}{e_z} D_z^D - \sum_z \frac{PL_z}{e_z} D_z^O + \sum_{zj} \frac{PMT_{zj}}{e_{zj}} IMT_{zj} - \sum_z \frac{PXT_z}{e_z} EXT_z$$

$$\sum_z \frac{PL_z}{e_z} XD_z + \sum_z \sum_{zj} PW_{z,zj}\ XM_{z,zj} =$$
$$\sum_z \frac{PL_z}{e_z} D_z^D - \sum_z \frac{PL_z}{e_z} D_z^O + \sum_z \frac{PMT_z}{e_z} IMT_z - \sum_z \frac{PXT_z}{e_z} EXT_z$$

1 We underline this as a precaution against the logical pitfall that would consist in using a discarded equation that could no longer be considered implicit in the model because the equations which imply it would also be absent from the model.

$$\sum_{z} \frac{PL_z}{e_z} XD_z + \sum_{z} \sum_{zj} PW_{z,zj} \ XM_{z,zj} =$$

$$\sum_{z} \left[\left(\frac{PL_z}{e_z} D_z^D + \frac{PMT_z}{e_z} IMT_z \right) - \left(\sum_{z} \frac{PL_z}{e_z} D_z^O + \sum_{z} \frac{PXT_z}{e_z} EXT_z \right) \right]$$

$$\sum_{z} \frac{PL_z}{e_z} XD_z + \sum_{z} \sum_{zj} PW_{z,zj} \ XM_{z,zj} =$$

$$\sum_{z} \frac{1}{e_z} \left[\left(PL_z D_z^D + PMT_z IMT_z \right) - \left(\sum_{z} PL_z D_z^O + \sum_{z} PXT_z EXT_z \right) \right]$$

Substitute (13) and (15) into the last equation, and

$$\sum_{z} \frac{PL_z}{e_z} XD_z + \sum_{z} \sum_{zj} PW_{z,zj} \ XM_{z,zj} = \sum_{z} \frac{1}{e_z} \left(PC_z Q_z - P_z XS_z \right) \tag{36}$$

where the regional agents' budget constraints are given by (3).

Regional budget constraints are homogeneous with respect to prices and nominal values[2]. Substitute the regional budget constraints into (36) to obtain

$$\sum_{z} \frac{PL_z}{e_z} XD_z + \sum_{z} \sum_{zj} PW_{z,zj} \ XM_{z,zj} = \sum_{z} \frac{1}{e_z} \left(- \ CAB_z \right)$$

Substitute (18) into the last equation (equation (18) is no longer present among the model equations, but implied: equations (13), (15) and (3) together imply (4), and equations (4) and (17) together imply (18)). The substitution yields

$$\sum_{z} \frac{PL_z}{e_z} XD_z + \sum_{z} \sum_{zj} PW_{z,zj} \ XM_{z,zj} = 0$$

So the total value of excess demands is zero, which is Walras' Law. It follows from Walras' Law that, if all excess demands but one are zero, then the remaining one is zero also. Therefore, one of the market equilibrium constraints consisting of (17) and $D_z^D = D_z^O$ is redundant (Of course, in the model, this equilibrium constraint is represented by the fact that both sides of the equation are one and the same variable, D_z).

---

2 In the traditional development of Walras' Law, the income and expenditures of each agent are constained to be equal, so that their budget constraints are homogeneous with respect to prices. Here, agents may have surpluses or deficits (non-zero CABs), the counterpart of which are, broadly speaking, international "loans". It follows that homogeneity must be defined not with respect to prices only, but with respect to prices *and* nominal values.

However, rather than discarding one of the market equilibrium constraints, it is possible to discard one of the budget constraints, while retaining all market equilibrium constraints. Indeed, if all excess demands are zero for some price vector, the left-hand side of equation (36) is zero. It follows that, for any region z, $z \in \{1, \cdots, N\}$,

$$\frac{1}{e_z}\left(PC_z Q_z - P_z XS_z\right) = -\sum_{zj \neq z} \frac{1}{e_{zj}}\left(PC_{zj} Q_{zj} - P_{zj} XS_{zj}\right)$$

while (18) implies

$$\frac{CAB_z}{e_z} = -\sum_{zj \neq z} \frac{CAB_{zj}}{e_{zj}}$$

Consequently, if $CAB_{zj} = P_{zj} XS_{zj} - PC_{zj} Q_{zj}$ for all $zj \neq z$, then the last two equations guarantee that, for the remaining region z also, $CAB_z = P_z XS_z - PC_z Q_z$. This is the form which Walras' Law takes in our model. We arbitrarily pick some region $zleon$, $zleon \in \{1, \cdots, N\}$ ($zleon$ is a mnemonic for Léon Walras), and remove equation (3) for that single region. Note that, with the removal of the equation relating to $zleon$, the variable CAB_{zleon} no longer appears in the model. Its value may be computed using the suppressed equation.

It is common practice in CGE modeling to introduce an extra variable and an extra equation to verify Walras' Law. In the GAMS implementation described in Appendix K, the extra variable is labeled $LEON$ in honor of Léon Walras, and $LEON = CAB_{zleon} - \left(P_{zleon}\ XS_{zleon} - PC_{zleon}\ Q_{zleon}\right)$. A nonzero $LEON$ in the solution indicates that there is an error in the model.

APPENDIX C: NLP PROBLEM WITH DUMMY OBJECTIVE

Consider the following example

$$x + y = a \tag{37}$$

$$bx + by = ba \tag{38}$$

$$y = c \tag{39}$$

where a, b and c are constants. Clearly, equation (38) is redundant given equation (37). Equation (39) plays the role of a closure rule.

Using the GAMS solver CONOPT, with the statement

```
Solve testmodel_1 using CNS;
```

where `testmodel_1` consists of equations (37), (38) and (39), the program aborts, because the model is not square: the CONOPT solver does not detect redundancy. On the other hand, with the statement

```
Solve testmodel_1 using NLP minimizing y;
```

the model finds a solution which is identical to the solution produced by the statement

```
Solve testmodel_2 using CNS;
```

where `testmodel_2` is the correct model, consisting of equations (37) and (39).

Now, suppose the modeler does not eliminate redundant equation (38), fails to include the (39) closure rule, and introduces a dummy objective variable, z. The resulting model is `testmodel_3`, which consists of equations (37) and (38), together with (40):

$$z = d \tag{40}$$

where d is any constant. We now perform

```
Solve testmodel_3 using NLP minimizing z;
```

Obviously, the solution of `testmodel_3` is not unique. The solution produced by the algorithm depends on the initial values of variables. If the `SOLVE` statement for `testmodel_3` comes immediately after the solution of `testmodel_2`, then y will be implicitly initialized at its correct solution value and the solution will be the same. But if the initial value of y is modified, then the solution will be different and, in our example, wrong.

APPENDIX D: EQUATION (32)

D.1 Cost minimizing problem

The regional agent allocates demand between domestic production and imports by minimizing consumption expenditures (equation (15)), subject to constraint (9).

D.2 Lagrangian and first-order conditions

Form the Lagrangian

$$\mathscr{L}_z = PL_z D_z + PMT_z \; IMT_z - \mu_z \left\{ A_z \left[\alpha_z \, D_z^{-\rho_z} + (1-\alpha_z) \, IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} - Q_z \right\}$$

The first order conditions are:

$$\frac{\partial \mathscr{L}_z}{\partial \mu_z} = -A_z \left[\alpha_z \, D_z^{-\rho_z} + (1-\alpha_z) \, IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} + Q_z = 0 \tag{41}$$

$$\frac{\partial \mathscr{L}_z}{\partial D_z} = PL_z - \mu_z \frac{\partial}{\partial D_z} A_z \left[\alpha_z \, D_z^{-\rho_z} + (1-\alpha_z) \, IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} = 0 \tag{42}$$

$$\frac{\partial \mathscr{L}_z}{\partial IMT_z} = PMT_z - \mu_z \frac{\partial}{\partial IMT_z} A_z \left[\alpha_z \, D_z^{-\rho_z} + (1-\alpha_z) \, IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} = 0 \tag{43}$$

with

$$\frac{\partial}{\partial D_z} A_z \left[\alpha_z \, D_z^{-\rho_z} + (1-\alpha_z) \, IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} =$$

$$- \frac{A_z}{\rho_z} \left[\alpha_z \, D_z^{-\rho_z} + (1-\alpha_z) \, IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}-1} \frac{\partial}{\partial D_z} \left[\alpha_z \, D_z^{-\rho_z} + (1-\alpha_z) \, IMT_z^{-\rho_z} \right]$$

$$\frac{\partial}{\partial D_z} A_z \left[\alpha_z \, D_z^{-\rho_z} + (1-\alpha_z) \, IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}} =$$
$$+ A_z \left[\alpha_z \, D_z^{-\rho_z} + (1-\alpha_z) \, IMT_z^{-\rho_z} \right]^{-\frac{\rho_z+1}{\rho_z}} \alpha_z D_z^{-\rho_z-1} \tag{44}$$

and

$$\frac{\partial}{\partial IMT_z} A_z\left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}} =$$

$$-\frac{A_z}{\rho_z}\left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}-1} \frac{\partial}{\partial IMT_z}\left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]$$

$$\frac{\partial}{\partial IMT_z} A_z\left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}} =$$

$$+ A_z\left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]^{-\frac{\rho_z+1}{\rho_z}} (1-\alpha_z) IMT_z^{-\rho_z-1} \tag{45}$$

Given (9),

$$\frac{Q_z}{A_z} = \left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}}$$

$$\left(\frac{Q_z}{A_z}\right)^{-\rho_z} = \left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]$$

Substitute the last equation into (44) to obtain

$$\frac{\partial}{\partial D_z} A_z\left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}} = A_z\left[\left(\frac{Q_z}{A_z}\right)^{-\rho_z}\right]^{-\frac{\rho_z+1}{\rho_z}} \alpha_z D_z^{-\rho_z-1}$$

$$\frac{\partial}{\partial D_z} A_z\left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}} = A_z\left(\frac{Q_z}{A_z}\right)^{\rho_z+1} \alpha_z D_z^{-\rho_z-1}$$

$$\frac{\partial}{\partial D_z} A_z\left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}} = A_z^{-\rho_z} \alpha_z\left(\frac{Q_z}{D_z}\right)^{\rho_z+1}$$

and similarly into (45), yielding

$$\frac{\partial}{\partial IMT_z} A_z\left[\alpha_z\, D_z^{-\rho_z} + (1-\alpha_z)\, IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}} = A_z^{-\rho_z} (1-\alpha_z)\left(\frac{Q_z}{IMT_z}\right)^{\rho_z+1}$$

First-order conditions (42) and (43) can now be written as

$$\frac{\partial \mathscr{L}_z}{\partial D_z} = PL_z - \mu_z A_z^{-\rho_z} \alpha_z \left(\frac{Q_z}{D_z}\right)^{\rho_z + 1} = 0$$

$$\frac{\partial \mathscr{L}_z}{\partial IMT_z} = PMT_z - \mu_z A_z^{-\rho_z} (1 - \alpha_z) \left(\frac{Q_z}{IMT_z}\right)^{\rho_z + 1} = 0$$

or

$$PL_z = \mu_z A_z^{-\rho_z} \alpha_z \left(\frac{Q_z}{D_z}\right)^{\rho_z + 1}$$

$$PMT_z = \mu_z A_z^{-\rho_z} (1 - \alpha_z) \left(\frac{Q_z}{IMT_z}\right)^{\rho_z + 1}$$

D.3 Relative demand for domestic production and imports

Take the ratio of the final two equations of D.2,

$$\frac{PL_z}{PMT_z} = \frac{\mu_z A_z^{-\rho_z} \alpha_z \left(\frac{Q_z}{D_z}\right)^{\rho_z + 1}}{\mu_z A_z^{-\rho_z} (1 - \alpha_z) \left(\frac{Q_z}{IMT_z}\right)^{\rho_z + 1}}$$

$$\frac{PL_z}{PMT_z} = \frac{\alpha_z IMT_z^{\rho_z + 1}}{(1 - \alpha_z) D_z^{\rho_z + 1}}$$

$$\left(\frac{IMT_z}{D_z}\right)^{\rho_z + 1} = \frac{(1 - \alpha_z) PL_z}{\alpha_z PMT_z}$$

$$\frac{IMT_z}{D_z} = \left(\frac{(1 - \alpha_z) PL_z}{\alpha_z PMT_z}\right)^{\frac{1}{\rho_z + 1}}$$

where $\rho_z = \dfrac{1 - \sigma_z}{\sigma_z}$ implies $\rho_z + 1 = \dfrac{1}{\sigma_z}$, and

$$\frac{IMT_z}{D_z} = \left(\frac{(1 - \alpha_z) PL_z}{\alpha_z PMT_z}\right)^{\sigma_z} \tag{46}$$

$$IMT_z = \left(\frac{(1 - \alpha_z) PL_z}{\alpha_z PMT_z}\right)^{\sigma_z} D_z$$

D.4 Demand for components in terms of composite demand and prices

Substitute the final equation of D.3 into (9)

$$Q_z = A_z \left\{ \alpha_z D_z^{-\rho_z} + (1-\alpha_z) \left[\left(\frac{(1-\alpha_z) PL_z}{\alpha_z PMT_z} \right)^{\sigma_z} D_z \right]^{-\rho_z} \right\}^{-\frac{1}{\rho_z}}$$

$$Q_z = A_z D_z \left\{ \alpha_z + (1-\alpha_z) \left[\left(\frac{(1-\alpha_z) PL_z}{\alpha_z PMT_z} \right)^{\sigma_z} \right]^{-\rho_z} \right\}^{-\frac{1}{\rho_z}}$$

where $\rho_z = \dfrac{1-\sigma_z}{\sigma_z}$ implies $\sigma_z = \dfrac{1}{\rho_z + 1}$, and

$$Q_z = A_z D_z \left\{ \alpha_z + (1-\alpha_z) \left(\frac{(1-\alpha_z) PL_z}{\alpha_z PMT_z} \right)^{-\frac{\rho_z}{\rho_z+1}} \right\}^{-\frac{1}{\rho_z}}$$

$$\frac{D_z}{Q_z} = \frac{1}{A_z} \left\{ \alpha_z + (1-\alpha_z) \left(\frac{(1-\alpha_z) PL_z}{\alpha_z PMT_z} \right)^{-\frac{\rho_z}{\rho_z+1}} \right\}^{\frac{1}{\rho_z}}$$

$$\frac{D_z}{Q_z} = \frac{1}{A_z} \left\{ \left(\frac{\alpha_z}{PL_z} \right)^{\frac{\rho_z}{\rho_z+1}} \left[\alpha_z \left(\frac{\alpha_z}{PL_z} \right)^{-\frac{\rho_z}{\rho_z+1}} + (1-\alpha_z) \left(\frac{(1-\alpha_z)}{PMT_z} \right)^{-\frac{\rho_z}{\rho_z+1}} \right] \right\}^{\frac{1}{\rho_z}}$$

$$\frac{D_z}{Q_z} = \frac{1}{A_z} \left(\frac{\alpha_z}{PL_z} \right)^{\frac{1}{\rho_z+1}} \left[\alpha_z \left(\frac{\alpha_z}{PL_z} \right)^{-\frac{\rho_z}{\rho_z+1}} + (1-\alpha_z) \left(\frac{(1-\alpha_z)}{PMT_z} \right)^{-\frac{\rho_z}{\rho_z+1}} \right]^{\frac{1}{\rho_z}}$$

where $\rho_z = \dfrac{1 - \sigma_z}{\sigma_z}$ implies $\rho_z + 1 = \dfrac{1}{\sigma_z}$, and $\dfrac{\rho_z}{\rho_z + 1} = 1 - \sigma_z$. It follows that

$$\frac{D_z}{Q_z} = \frac{1}{A_z}\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z}\left[\alpha_z\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z - 1} + (1 - \alpha_z)\left(\frac{(1 - \alpha_z)}{PMT_z}\right)^{\sigma_z - 1}\right]^{-\frac{\sigma_z}{\sigma_z - 1}}$$

$$\frac{D_z}{Q_z} = \frac{1}{A_z}\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z}\left[PL_z\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z} + PMT_z\left(\frac{(1 - \alpha_z)}{PMT_z}\right)^{\sigma_z}\right]^{-\frac{\sigma_z}{\sigma_z - 1}} \tag{47}$$

where the left-hand side ratio depends only on component prices. As a matter of fact, multiplying both sides of (47) by Q_z transforms it into a demand equation of D_z. A similar development would lead to a demand equation for IMT_z.

$$\frac{IMT_z}{Q_z} = \frac{1}{A_z}\left(\frac{(1 - \alpha_z)}{PMT_z}\right)^{\sigma_z}\left[PL_z\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z} + PMT_z\left(\frac{(1 - \alpha_z)}{PMT_z}\right)^{\sigma_z}\right]^{-\frac{\sigma_z}{\sigma_z - 1}}$$

D.5 Unit cost of composite demand

It is clear from (46) that the relative demand for domestic production and imports is independent of the scale of demand, which is consistent with the first-degree homogeneity of aggregator function (9). Consequently, the unit cost of the composite demand can be obtained from (15) by substituting the optimal ratio determined in (46), and dividing through by Q_z.

$$PC_zQ_z = \left(PL_z + PMT_z\frac{IMT_z}{D_z}\right)D_z$$

$$PC_zQ_z = \left[PL_z + PMT_z\left(\frac{(1 - \alpha_z)PL_z}{\alpha_zPMT_z}\right)^{\sigma_z}\right]D_z$$

$$PC_z = \left[PL_z + PMT_z\left(\frac{(1 - \alpha_z)PL_z}{\alpha_zPMT_z}\right)^{\sigma_z}\right]\frac{D_z}{Q_z}$$

$$PC_z = \left(\frac{PL_z}{\alpha_z}\right)^{\sigma_z}\left[PL_z\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z} + PMT_z\left(\frac{(1 - \alpha_z)}{PMT_z}\right)^{\sigma_z}\right]\frac{D_z}{Q_z}$$

To express PC_z in terms of the component prices only, we must substitute for D_z/Q_z in this last equation using (47):

$$PC_z = \left(\frac{PL_z}{\alpha_z}\right)^{\sigma_z}\left[PL_z\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z} + PMT_z\left(\frac{(1-\alpha_z)}{PMT_z}\right)^{\sigma_z}\right]$$

$$\frac{1}{A_z}\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z}\left[PL_z\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z} + PMT_z\left(\frac{(1-\alpha_z)}{PMT_z}\right)^{\sigma_z}\right]^{\frac{\sigma_z}{\sigma_z-1}}$$

$$PC_z = \frac{1}{A_z}\left[PL_z\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z} + PMT_z\left(\frac{(1-\alpha_z)}{PMT_z}\right)^{\sigma_z}\right]^{1+\frac{\sigma_z}{\sigma_z-1}}$$

$$PC_z = \frac{1}{A_z}\left[PL_z\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z} + PMT_z\left(\frac{(1-\alpha_z)}{PMT_z}\right)^{\sigma_z}\right]^{-\frac{1}{\sigma_z-1}}$$

D.6 Demand for domestic production and imports reformulated

Given the last equation in D.5, we have

$$A_zPC_z = \left[PL_z\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z} + PMT_z\left(\frac{(1-\alpha_z)}{PMT_z}\right)^{\sigma_z}\right]^{-\frac{1}{\sigma_z-1}}$$

Using the preceding equation, we can write (47) as

$$\frac{D_z}{Q_z} = \frac{1}{A_z}\left(\frac{\alpha_z}{PL_z}\right)^{\sigma_z}(A_zPC_z)^{\sigma_z}$$

$$\frac{D_z}{Q_z} = (A_z)^{\sigma_z-1}\left(\frac{\alpha_zPC_z}{PL_z}\right)^{\sigma_z}$$

$$D_z = (A_z)^{\sigma_z-1}\left(\frac{\alpha_zPC_z}{PL_z}\right)^{\sigma_z}Q_z \qquad (48)$$

which is an alternate form of the demand equation for domestic production. A similar development leads to the import demand equation:

$$IMT_z = (A_z)^{\sigma_z - 1} \left(\frac{(1 - \alpha_z)PC_z}{PMT_z} \right)^{\sigma_z} Q_z \tag{49}$$

Equation (32) defines D_z^D from (48).

APPENDIX E: EQUATION (33)

E.1 Sales revenue maximizing problem

The regional agent allocates production between the domestic market and exports by maximizing the value of production (equation (13)), subject to constraint (5).

E.2 Lagrangian and first-order conditions

Form the Lagrangian

$$\mathscr{L}_z = PL_z D_z + PXT_z \ EXT - \lambda_z \left\{ B_z \left[\beta_z \ D_z^{\kappa_z} + (1-\beta_z) \ EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} - XS_z \right\}$$

The first order conditions are:

$$\frac{\partial \mathscr{L}_z}{\partial \lambda_z} = -B_z \left[\beta_z \ D_z^{\kappa_z} + (1-\beta_z) \ EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} + XS_z = 0 \tag{50}$$

$$\frac{\partial \mathscr{L}_z}{\partial D_z} = PL_z - \lambda_z \frac{\partial}{\partial D_z} B_z \left[\beta_z \ D_z^{\kappa_z} + (1-\beta_z) \ EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} = 0 \tag{51}$$

$$\frac{\partial \mathscr{L}_z}{\partial EXT_z} = PXT_z - \lambda_z \frac{\partial}{\partial EXT_z} B_z \left[\beta_z \ D_z^{\kappa_z} + (1-\beta_z) \ EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} = 0 \tag{52}$$

with

$$\frac{\partial}{\partial D_z} B_z \left[\beta_z \ D_z^{\kappa_z} + (1-\beta_z) \ EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} =$$

$$\frac{B_z}{\kappa_z} \left[\beta_z \ D_z^{\kappa_z} + (1-\beta_z) \ EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}-1} \frac{\partial}{\partial D_z} \left[\beta_z \ D_z^{\kappa_z} + (1-\beta_z) \ EXT_z^{\kappa_z} \right]$$

$$\begin{aligned} &\frac{\partial}{\partial D_z} B_z \left[\beta_z \ D_z^{\kappa_z} + (1-\beta_z) \ EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}} = \\ &\qquad B_z \left[\beta_z \ D_z^{\kappa_z} + (1-\beta_z) \ EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}-1} \beta_z \ D_z^{\kappa_z - 1} \end{aligned} \tag{53}$$

and

$$\frac{\partial}{\partial EXT_z} B_z\left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]^{\frac{1}{\kappa_z}} =$$

$$\frac{B_z}{\kappa_z}\left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]^{\frac{1}{\kappa_z}-1} \frac{\partial}{\partial EXT_z}\left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]$$

$$\frac{\partial}{\partial EXT_z} B_z\left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]^{\frac{1}{\kappa_z}} =$$

$$B_z\left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]^{\frac{1}{\kappa_z}-1} (1-\beta_z)\, EXT_z^{\kappa_z-1} \tag{54}$$

Given (5),

$$\frac{XS_z}{B_z} = \left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]^{\frac{1}{\kappa_z}}$$

$$\left(\frac{XS_z}{B_z}\right)^{\kappa_z} = \left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]$$

Substitute that into (53) to obtain

$$\frac{\partial}{\partial D_z} B_z\left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]^{\frac{1}{\kappa_z}} = B_z\left[\left(\frac{XS_z}{B_z}\right)^{\kappa_z}\right]^{\frac{1}{\kappa_z}-1} \beta_z\, D_z^{\kappa_z-1}$$

$$\frac{\partial}{\partial D_z} B_z\left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]^{\frac{1}{\kappa_z}} = B_z^{\kappa_z} XS_z^{1-\kappa_z} \beta_z\, D_z^{\kappa_z-1}$$

$$\frac{\partial}{\partial D_z} B_z\left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]^{\frac{1}{\kappa_z}} = B_z^{\kappa_z} \beta_z \left(\frac{D_z}{XS_z}\right)^{\kappa_z-1}$$

and similarly into (54), which yields

$$\frac{\partial}{\partial EXT_z} B_z\left[\beta_z\, D_z^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z}\right]^{\frac{1}{\kappa_z}} = B_z^{\kappa_z}(1-\beta_z)\left(\frac{EXT_z}{XS_z}\right)^{\kappa_z-1}$$

First-order conditions (51) and (52) can now be written as

$$\frac{\partial \mathscr{L}_z}{\partial D_z} = PL_z - \lambda_z B_z^{\kappa_z} \beta_z \left(\frac{D_z}{XS_z} \right)^{\kappa_z - 1} = 0$$

$$\frac{\partial \mathscr{L}_z}{\partial EXT_z} = PXT_z - \lambda_z B_z^{\kappa_z} (1 - \beta_z) \left(\frac{EXT_z}{XS_z} \right)^{\kappa_z - 1} = 0$$

or

$$PL_z = \lambda_z B_z^{\kappa_z} \beta_z \left(\frac{D_z}{XS_z} \right)^{\kappa_z - 1}$$

$$PXT_z = \lambda_z B_z^{\kappa_z} (1 - \beta_z) \left(\frac{EXT_z}{XS_z} \right)^{\kappa_z - 1}$$

E.3 Relative supply of components

Take the ratio of the two last equations in E.2:

$$\frac{PL_z}{PXT_z} = \frac{\lambda_z B_z^{\kappa_z} \beta_z \left(\frac{D_z}{XS_z} \right)^{\kappa_z - 1}}{\lambda_z B_z^{\kappa_z} (1 - \beta_z) \left(\frac{EXT_z}{XS_z} \right)^{\kappa_z - 1}}$$

$$\frac{PL_z}{PXT_z} = \frac{\beta_z \, (D_z)^{\kappa_z - 1}}{(1 - \beta_z) \, (EXT_z)^{\kappa_z - 1}}$$

$$\frac{(D_z)^{\kappa_z - 1}}{(EXT_z)^{\kappa_z - 1}} = \frac{(1 - \beta_z) \, PL_z}{\beta_z \, PXT_z}$$

$$\frac{D_z}{EXT_z} = \left(\frac{(1 - \beta_z) \, PL_z}{\beta_z \, PXT_z} \right)^{\frac{1}{\kappa_z - 1}}$$

where $\kappa_z = \frac{\tau_z + 1}{\tau_z}$ implies $\kappa_z - 1 = \frac{1}{\tau_z}$ and

$$\frac{D_z}{EXT_z} = \left(\frac{(1 - \beta_z) \, PL_z}{\beta_z \, PXT_z} \right)^{\tau_z} \tag{55}$$

$$D_z = \left(\frac{(1 - \beta_z) \, PL_z}{\beta_z \, PXT_z} \right)^{\tau_z} EXT_z$$

E.4 Supply of components in terms of aggregate production and prices

Substitute the last equation of E.3 into (5)

$$XS_z = B_z \left\{ \beta_z \left[\left(\frac{(1-\beta_z)\, PL_z}{\beta_z\, PXT_z} \right)^{\tau_z} EXT_z \right]^{\kappa_z} + (1-\beta_z)\, EXT_z^{\kappa_z} \right\}^{\frac{1}{\kappa_z}}$$

$$XS_z = B_z EXT_z \left\{ \beta_z \left[\left(\frac{(1-\beta_z)\, PL_z}{\beta_z\, PXT_z} \right)^{\tau_z} \right]^{\kappa_z} + (1-\beta_z) \right\}^{\frac{1}{\kappa_z}}$$

where $\kappa_z = \dfrac{\tau_z + 1}{\tau_z}$ implies $\tau_z = \dfrac{1}{\kappa_z - 1}$ and

$$XS_z = B_z EXT_z \left\{ \beta_z \left[\left(\frac{(1-\beta_z)\, PL_z}{\beta_z\, PXT_z} \right)^{\frac{1}{\kappa_z - 1}} \right]^{\kappa_z} + (1-\beta_z) \right\}^{\frac{1}{\kappa_z}}$$

$$XS_z = B_z EXT_z \left\{ \beta_z \left(\frac{(1-\beta_z)\, PL_z}{\beta_z\, PXT_z} \right)^{\frac{\kappa_z}{\kappa_z - 1}} + (1-\beta_z) \right\}^{\frac{1}{\kappa_z}}$$

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z} \left\{ \beta_z \left(\frac{(1-\beta_z)\, PL_z}{\beta_z\, PXT_z} \right)^{\frac{\kappa_z}{\kappa_z - 1}} + (1-\beta_z) \right\}^{-\frac{1}{\kappa_z}}$$

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z} \left\{ \left(\frac{(1-\beta_z)}{PXT_z} \right)^{\frac{\kappa_z}{\kappa_z - 1}} \left[\beta_z \left(\frac{PL_z}{\beta_z} \right)^{\frac{\kappa_z}{\kappa_z - 1}} + (1-\beta_z) \left(\frac{PXT_z}{(1-\beta_z)} \right)^{\frac{\kappa_z}{\kappa_z - 1}} \right] \right\}^{-\frac{1}{\kappa_z}}$$

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z}\left(\frac{PXT_z}{(1-\beta_z)}\right)^{\frac{1}{\kappa_z - 1}}\left[\beta_z\left(\frac{PL_z}{\beta_z}\right)^{\frac{\kappa_z}{\kappa_z - 1}} + (1-\beta_z)\left(\frac{PXT_z}{(1-\beta_z)}\right)^{\frac{\kappa_z}{\kappa_z - 1}}\right]^{-\frac{1}{\kappa_z}}$$

where $\kappa_z = \dfrac{\tau_z + 1}{\tau_z}$ implies $\kappa_z - 1 = \dfrac{1}{\tau_z}$ and $\dfrac{\kappa_z}{\kappa_z - 1} = \tau_z + 1$. It follows that

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z}\left(\frac{PXT_z}{(1-\beta_z)}\right)^{\tau_z}\left[\beta_z\left(\frac{PL_z}{\beta_z}\right)^{\tau_z + 1} + (1-\beta_z)\left(\frac{PXT_z}{(1-\beta_z)}\right)^{\tau_z + 1}\right]^{-\frac{\tau_z}{\tau_z + 1}}$$

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z}\left(\frac{PXT_z}{(1-\beta_z)}\right)^{\tau_z}\left[PL_z\left(\frac{PL_z}{\beta_z}\right)^{\tau_z} + PXT_z\left(\frac{PXT_z}{(1-\beta_z)}\right)^{\tau_z}\right]^{-\frac{\tau_z}{\tau_z + 1}} \tag{56}$$

where the left-hand side ratio depends only on component prices. As a matter of fact, multiplying both sides of (56) by XS_z transforms it into a supply equation of EXT_z. A similar development would lead to a supply equation for D_z.

$$\frac{D_z}{XS_z} = \frac{1}{B_z}\left(\frac{PL_z}{\beta_z}\right)^{\tau_z}\left[PL_z\left(\frac{PL_z}{\beta_z}\right)^{\tau_z} + PXT_z\left(\frac{PXT_z}{(1-\beta_z)}\right)^{\tau_z}\right]^{-\frac{\tau_z}{\tau_z + 1}}$$

E.5 Unit value of aggregate output

It is clear from (55) that the relative supply on the domestic and export markets is independent of the scale of output, consistent with the first-degree homogeneity of aggregator function (5). Consequently, the unit value of the output aggregate can be obtained from (13) by substituting the optimal ratio determined in (55), and dividing through by XS_z.

$$P_z\, XS_z = \left(PL_z \frac{D_z}{EXT_z} + PXT_z\right) EXT_z$$

$$P_z\, XS_z = \left[PL_z\left(\frac{(1-\beta_z)\, PL_z}{\beta_z\, PXT_z}\right)^{\tau_z} + PXT_z\right] EXT_z$$

$$P_z = \left[PL_z \left(\frac{(1-\beta_z)\, PL_z}{\beta_z\, PXT_z} \right)^{\tau_z} + PXT_z \right] \frac{EXT_z}{XS_z}$$

$$P_z = \left(\frac{(1-\beta_z)}{PXT_z} \right)^{\tau_z} \left[PL_z \left(\frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left(\frac{PXT_z}{(1-\beta_z)} \right)^{\tau_z} \right] \frac{EXT_z}{XS_z}$$

To express P_z in terms of the component prices only, we must substitute for EXT_z / XS_z in the preceding equation using (56):

$$P_z = \left(\frac{(1-\beta_z)}{PXT_z} \right)^{\tau_z} \left[PL_z \left(\frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left(\frac{PXT_z}{(1-\beta_z)} \right)^{\tau_z} \right]$$

$$\frac{1}{B_z} \left(\frac{PXT_z}{(1-\beta_z)} \right)^{\tau_z} \left[PL_z \left(\frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left(\frac{PXT_z}{(1-\beta_z)} \right)^{\tau_z} \right]^{-\frac{\tau_z}{\tau_z+1}}$$

$$P_z = \frac{1}{B_z} \left[PL_z \left(\frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left(\frac{PXT_z}{(1-\beta_z)} \right)^{\tau_z} \right] \left[PL_z \left(\frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left(\frac{PXT_z}{(1-\beta_z)} \right)^{\tau_z} \right]^{-\frac{\tau_z}{\tau_z+1}}$$

$$P_z = \frac{1}{B_z} \left[PL_z \left(\frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left(\frac{PXT_z}{(1-\beta_z)} \right)^{\tau_z} \right]^{\frac{1}{\tau_z+1}}$$

E.6 Supply of components reformulated

Given the last equation in E.5, we have

$$B_z P_z = \left[PL_z \left(\frac{PL_z}{\beta_z} \right)^{\tau_z} + PXT_z \left(\frac{PXT_z}{(1-\beta_z)} \right)^{\tau_z} \right]^{\frac{1}{\tau_z+1}}$$

Using this, we can write (56) as

$$\frac{EXT_z}{XS_z} = \frac{1}{B_z} \left(\frac{PXT_z}{(1-\beta_z)} \right)^{\tau_z} (B_z P_z)^{-\tau_z}$$

$$\frac{EXT_z}{XS_z} = \left(\frac{1}{B_z} \right)^{\tau_z+1} \left(\frac{PXT_z}{(1-\beta_z) P_z} \right)^{\tau_z}$$

$$EXT_z = \left(\frac{1}{B_z}\right)^{\tau_z+1}\left(\frac{PXT_z}{(1-\beta_z)P_z}\right)^{\tau_z} XS_z \tag{57}$$

which is an alternate form of the export supply equation. A similar development leads to the domestic supply equation:

$$D_z = \left(\frac{1}{B_z}\right)^{\tau_z+1}\left(\frac{PL_z}{\beta_z P_z}\right)^{\tau_z} XS_z \tag{58}$$

Equation (33) defines D_z^O from (58).

APPENDIX F: EQUATION (34)

F.1 Cost minimizing problem

The regional agent allocates imports between origins by minimizing the aggregate cost of imports (equation (16)) subject to constraint (11) and to $IMT_z = IMT_z^*$, where IMT_z^* is the solution to the demand allocation problem as given by (49).

F.2 Lagrangian and first-order conditions

Write the Lagrangian

$$\mathscr{L}_z = \sum_{zj} PW_{zj,z} IM_{zj,z} - \mu_z^M \left\{ A_z^M \left[\sum_{zj} \alpha_{zj,z}^M \, IM_{zj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} - IMT_z^* \right\}$$

The first order conditions are:

$$\frac{\partial \mathscr{L}_z}{\partial \mu_z^X} = -A_z^M \left[\sum_{zj} \alpha_{zj,z}^M \, IM_{zj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} + IMT_z^* = 0$$

$$\frac{\partial \mathscr{L}_z}{\partial IM_{zj,z}} = PW_{zj,z} - \mu_z^M \frac{\partial}{\partial IM_{zj,z}} A_z^M \left[\sum_{zjj} \alpha_{zjj,z}^M \, IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} = 0$$

where

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[\sum_{zjj} \alpha_{zjj,z}^M \, IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} =$$

$$- A_z^M \frac{1}{\rho_z^M} \left[\sum_{zjj} \alpha_{zjj,z}^M \, IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M} - 1} \frac{\partial}{\partial IM_{zj,z}} \left[\sum_{zjj} \alpha_{zjj,z}^M \, IM_{zjj,z}^{-\rho_z^M} \right]$$

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[\sum_{zjj} \alpha_{zjj,z}^M \ IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} =$$

$$- \ A_z^M \ \frac{1}{\rho_z^M} \left[\sum_{zjj} \alpha_{zjj,z}^M \ IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}-1} \alpha_{zj,z}^M \left(- \ \rho_z^M \right) \ IM_{zj,z}^{-\rho_z^M-1}$$

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[\sum_{zjj} \alpha_{zjj,z}^M \ IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} = A_z^M \left[\sum_{zjj} \alpha_{zjj,z}^M \ IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}-1} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M-1}$$

Given (11), which is equivalent to

$$\left(\frac{IMT_z}{A_z^M} \right)^{-\rho_z^M} = \left[\sum_{zj} \alpha_{zj,z}^M \ IM_{zj,z}^{-\rho_z^M} \right]$$

we can write

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[\sum_{zjj} \alpha_{zjj,z}^M \ IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} = A_z^M \left[\left(\frac{IMT_z^*}{A_z^M} \right)^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}-1} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M-1}$$

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[\sum_{zjj} \alpha_{zjj,z}^M \ IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} = A_z^M \left(\frac{IMT_z^*}{A_z^M} \right)^{\rho_z^M+1} \alpha_{zj,z}^M IM_{zj,z}^{-\rho_z^M-1}$$

$$\frac{\partial}{\partial IM_{zj,z}} A_z^M \left[\sum_{zjj} \alpha_{zjj,z}^M \ IM_{zjj,z}^{-\rho_z^M} \right]^{\frac{-1}{\rho_z^M}} = \left(A_z^M \right)^{-\rho_z^M} \alpha_{zj,z}^M \left(\frac{IMT_z^*}{IM_{zj,z}} \right)^{\rho_z^M+1}$$

And the first-order condition becomes

$$\frac{\partial \mathscr{L}_z}{\partial IM_{zj,z}} = PW_{zj,z} - \mu_z^M \left(A_z^M \right)^{-\rho_z^M} \alpha_{zj,z}^M \left(\frac{IMT_z^*}{IM_{zj,z}} \right)^{\rho_z^M+1} = 0$$

or

$$PW_{zj,z} = \mu_z^M \left(A_z^M \right)^{-\rho_z^M} \alpha_{zj,z}^M \left(\frac{IMT_z^*}{IM_{zj,z}} \right)^{\rho_z^M+1}$$

F.3 Relative demand for imports of different origins

Take the ratio of the preceding equation for two different origins, *zj* and *zjj*:

$$\frac{PW_{zjj,z}}{PW_{zj,z}} = \frac{\mu_z^M \left(A_z^M\right)^{-\rho_z^M} \alpha_{zjj,z}^M \left(\frac{IMT_z^*}{IM_{zjj,z}}\right)^{\rho_z^M+1}}{\mu_z^M \left(A_z^M\right)^{-\rho_z^M} \alpha_{zj,z}^M \left(\frac{IMT_z^*}{IM_{zj,z}}\right)^{\rho_z^M+1}}$$

$$\frac{PW_{zjj,z}}{PW_{zj,z}} = \frac{\alpha_{zjj,z}^M}{\alpha_{zj,z}^M}\left(\frac{IM_{zj,z}}{IM_{zjj,z}}\right)^{\rho_z^M+1}$$

$$\frac{IM_{zj,z}}{IM_{zjj,z}} = \left(\frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}}\right)^{\frac{1}{\rho_z^M+1}}$$

or, given $\rho_z^M = \frac{1-\sigma_z^M}{\sigma_z^M}$ and $\rho_z^M + 1 = \frac{1}{\sigma_z^M}$,

$$\frac{IM_{zj,z}}{IM_{zjj,z}} = \left(\frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}}\right)^{\sigma_z^M} \tag{59}$$

$$IM_{zj,z} = \left(\frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}}\right)^{\sigma_z^M} IM_{zjj,z}$$

F.4 Demand for components in terms of aggregate imports and prices

Substitute the last equation of F.3 into (11), and

$$IMT_z^* = A_z^M \left\{ \sum_{zj} \alpha_{zj,z}^M \left[\left(\frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}}\right)^{\sigma_z^M} IM_{zjj,z} \right]^{-\rho_z^M} \right\}^{\frac{-1}{\rho_z^M}}$$

where $\rho_z^M = \frac{1-\sigma_z^M}{\sigma_z^M}$ implies $\sigma_z^M = \frac{1}{\rho_z^M + 1}$. Then

$$IMT_z^* = A_z^M \left\{ \sum_{zj} \alpha_{zj,z}^M \left[\left(\frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{\frac{1}{\rho_z^M + 1}} IM_{zjj,z} \right]^{-\rho_z^M} \right\}^{\frac{-1}{\rho_z^M}}$$

$$IMT_z^* = A_z^M IM_{zjj,z} \left\{ \sum_{zj} \alpha_{zj,z}^M \left[\left(\frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{\frac{1}{\rho_z^M + 1}} \right]^{-\rho_z^M} \right\}^{\frac{-1}{\rho_z^M}}$$

$$IMT_z^* = A_z^M IM_{zjj,z} \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}} \right)^{-\frac{\rho_z^M}{\rho_z^M + 1}} \right]^{\frac{-1}{\rho_z^M}}$$

$$IMT_z^* = A_z^M IM_{zjj,z} \left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{\frac{1}{\rho_z^M + 1}} \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{-\frac{\rho_z^M}{\rho_z^M + 1}} \right]^{\frac{-1}{\rho_z^M}}$$

$$\frac{IM_{zjj,z}}{IMT_z^*} = \frac{1}{A_z^M} \left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{-\frac{1}{\rho_z^M + 1}} \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{-\frac{\rho_z^M}{\rho_z^M + 1}} \right]^{\frac{1}{\rho_z^M}}$$

where $\rho_z^M = \frac{1 - \sigma_z^M}{\sigma_z^M}$ implies $\rho_z^M + 1 = \frac{1}{\sigma_z^M}$, and $\frac{\rho_z^M}{\rho_z^M + 1} = 1 - \sigma_z^M$. Therefore

$$\frac{IM_{zjj,z}}{IMT_z^*} = \frac{1}{A_z^M} \left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M} \right)^{-\sigma_z^M} \left[\sum_{zj} \alpha_{zj,z}^M \left(\frac{\alpha_{zj,z}^M}{PW_{zj,z}} \right)^{\sigma_z^M - 1} \right]^{-\frac{\sigma_z^M}{\sigma_z^M - 1}}$$

$$\frac{IM_{zjj,z}}{IMT_z^*} = \frac{1}{A_z^M}\left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M}\right)^{-\sigma_z^M}\left[\sum_{zj} PW_{zj,z}\left(\frac{\alpha_{zj,z}^M}{PW_{zj,z}}\right)^{\sigma_z^M}\right]^{-\frac{\sigma_z^M}{\sigma_z^M-1}} \tag{60}$$

where the left-hand side ratio depends only on component prices. As a matter of fact, multiplying both sides of (60) by IMT_z^* transforms it into a supply equation of $IM_{zj,z}$.

F.5 Unit value of aggregate imports

It is clear from (59) that the relative demand for imports of different origins is independent of the scale of imports, consistent with the first-degree homogeneity of aggregator function (7). Consequently, the unit value of the import aggregate can be obtained from (16) by substituting the optimal ratio determined in (59), and dividing through by EXT_z^*.

$$PMT_z \;\; IMT_z^* = e_z IM_{zjj,z}\sum_{zj} PW_{zj,z}\left(\frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}}\right)^{\sigma_z^M}$$

$$PMT_z = e_z \frac{IM_{zjj,z}}{IMT_z^*}\sum_{zj} PW_{zj,z}\left(\frac{\alpha_{zj,z}^M PW_{zjj,z}}{\alpha_{zjj,z}^M PW_{zj,z}}\right)^{\sigma_z^M}$$

$$PMT_z = e_z \frac{IM_{zjj,z}}{IMT_z^*}\left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M}\right)^{\sigma_z^M}\sum_{zj} PW_{zj,z}\left(\frac{\alpha_{zj,z}^M}{PW_{zj,z}}\right)^{\sigma_z^M}$$

To express PMT_z in terms of the component prices only, we must substitute for $IM_{zjj,z}\big/IMT_z^*$ in the preceding equation using (60):

$$PMT_z = e_z \frac{1}{A_z^M}\left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M}\right)^{-\sigma_z^M}\left[\sum_{zj} PW_{zj,z}\left(\frac{\alpha_{zj,z}^M}{PW_{zj,z}}\right)^{\sigma_z^M}\right]^{-\frac{\sigma_z^M}{\sigma_z^M-1}}$$

$$\left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M}\right)^{\sigma_z^M}\sum_{zj} PW_{zj,z}\left(\frac{\alpha_{zj,z}^M}{PW_{zj,z}}\right)^{\sigma_z^M}$$

$$PMT_z = e_z \frac{1}{A_z^M}\left[\sum_{zj} PW_{zj,z}\left(\frac{\alpha_{zj,z}^M}{PW_{zj,z}}\right)^{\sigma_z^M}\right]^{1-\frac{\sigma_z^M}{\sigma_z^M-1}}$$

$$PMT_z = e_z \frac{1}{A_z^M}\left[\sum_{zj} PW_{zj,z}\left(\frac{\alpha_{zj,z}^M}{PW_{zj,z}}\right)^{\sigma_z^M}\right]^{-\frac{1}{\sigma_z^M-1}}$$

F.6 Supply of components reformulated

Given the preceding equation, we have

$$A_z^M PMT_z = e_z\left[\sum_{zj} PW_{zj,z}\left(\frac{\alpha_{zj,z}^M}{PW_{zj,z}}\right)^{\sigma_z^M}\right]^{-\frac{1}{\sigma_z^M-1}}$$

Using that equation, we can rewrite (60) as

$$\frac{IM_{zjj,z}}{IMT_z^*} = \frac{1}{A_z^M}\left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M}\right)^{-\sigma_z^M}\left(A_z^M PMT_z\right)^{\sigma_z^M}$$

$$\frac{IM_{zjj,z}}{IMT_z^*} = \left(\frac{1}{A_z^M}\right)^{1-\sigma_z^M}\left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M PMT_z}\right)^{-\sigma_z^M}$$

$$IM_{zjj,z} = \left(\frac{1}{A_z^M}\right)^{1-\sigma_z^M}\left(\frac{PW_{zjj,z}}{\alpha_{zjj,z}^M PMT_z}\right)^{-\sigma_z^M} IMT_z^*$$

which is an alternate form of the demand equation for imports from a particular region.

To obtain equation (34), substitute (49) into (61).

APPENDIX G: EQUATION (35)

G.1 Revenue maximizing problem

The regional agent allocates exports between export destinations by maximizing the value of exports (equation (14)), subject to constraint (7) and to $EXT_z = EXT_z^*$, where EXT_z^* is the solution to the output allocation problem as given by equation (57).

G.2 Lagrangian and first-order conditions

Write the Lagrangian

$$\mathscr{L}_z = \sum_{zj} PW_{z,zj} EX_{z,zj} - \lambda_z^X \left\{ B_z^X \left[\sum_{zj} \beta_{z,zj}^X \left(EX_{z,zj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} - EXT_z^* \right\}$$

The first order conditions are:

$$\frac{\partial \mathscr{L}_z}{\partial \lambda_z^X} = -B_z^X \left[\sum_{zj} \beta_{z,zj}^X \left(EX_{z,zj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} + EXT_z^* = 0$$

$$\frac{\partial \mathscr{L}_z}{\partial EX_{z,zj}} = PW_{z,zj} - \lambda_z^X \frac{\partial}{\partial EX_{z,zj}} B_z^X \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} = 0$$

where

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} =$$

$$B_z^X \frac{1}{\kappa_z^X} \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}-1} \frac{\partial}{\partial EX_{z,zj}} \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]$$

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} =$$

$$B_z^X \frac{1}{\kappa_z^X} \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}-1} \beta_{z,zj}^X \kappa_z^X \left(EX_{z,zj} \right)^{\kappa_z^X-1}$$

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} =$$

$$B_z^X \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}-1} \beta_{z,zj}^X \left(EX_{z,zj} \right)^{\kappa_z^X-1}$$

Given (7), which is equivalent to

$$\left(\frac{EXT_z}{B_z^X} \right)^{\kappa_z} = \left[\sum_{zj} \beta_{z,zj}^X \left(EX_{z,zj} \right)^{\kappa_z^X} \right]$$

we can write

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} = B_z^X \left[\left(\frac{EXT_z^*}{B_z^X} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}-1} \beta_{z,zj}^X \left(EX_{z,zj} \right)^{\kappa_z^X-1}$$

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} = B_z^X \left(\frac{EXT_z^*}{B_z^X} \right)^{1-\kappa_z^X} \beta_{z,zj}^X \left(EX_{z,zj} \right)^{\kappa_z^X-1}$$

$$\frac{\partial}{\partial EX_{z,zj}} B_z^X \left[\sum_{zjj} \beta_{z,zjj}^X \left(EX_{z,zjj} \right)^{\kappa_z^X} \right]^{\frac{1}{\kappa_z^X}} = \left(B_z^X \right)^{\kappa_z^X} \beta_{z,zj}^X \left(\frac{EX_{z,zj}}{EXT_z^*} \right)^{\kappa_z^X-1}$$

And the first-order condition becomes

$$\frac{\partial \mathscr{L}_z}{\partial EX_{z,zj}} = PW_{z,zj} - \lambda_z^X \left(B_z^X \right)^{\kappa_z^X} \beta_{z,zj}^X \left(\frac{EX_{z,zj}}{EXT_z^*} \right)^{\kappa_z^X-1} = 0$$

or

$$PW_{z,zj} = \lambda_z^X \left(B_z^X\right)^{\kappa_z^X} \beta_{z,zj}^X \left(\frac{EX_{z,zj}}{EXT_z^*}\right)^{\kappa_z^X - 1}$$

G.3 Relative supply of components

Take the ratio of the preceding equation for two different trading partners of region *z*: *zj* and *zjj*.

$$\frac{PW_{z,zj}}{PW_{z,zjj}} = \frac{\lambda_z^X \left(B_z^X\right)^{\kappa_z^X} \beta_{z,zj}^X \left(\dfrac{EX_{z,zj}}{EXT_z^*}\right)^{\kappa_z^X - 1}}{\lambda_z^X \left(B_z^X\right)^{\kappa_z^X} \beta_{z,zjj}^X \left(\dfrac{EX_{z,zjj}}{EXT_z^*}\right)^{\kappa_z^X - 1}}$$

$$\frac{PW_{z,zj}}{PW_{z,zjj}} = \frac{\beta_{z,zj}^X}{\beta_{z,zjj}^X} \left(\frac{EX_{z,zj}}{EX_{z,zjj}}\right)^{\kappa_z^X - 1}$$

$$\left(\frac{EX_{z,zj}}{EX_{z,zjj}}\right)^{\kappa_z^X - 1} = \frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}}$$

$$\frac{EX_{z,zj}}{EX_{z,zjj}} = \left(\frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}}\right)^{\frac{1}{\kappa_z^X - 1}}$$

Or, given $\kappa_z^X = \dfrac{\tau_z^X + 1}{\tau_z^X}$ and $\kappa_z^X - 1 = \dfrac{1}{\tau_z^X}$,

$$\frac{EX_{z,zj}}{EX_{z,zjj}} = \left(\frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}}\right)^{\tau_z^X} \tag{62}$$

$$EX_{z,zj} = \left(\frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}}\right)^{\tau_z^X} EX_{z,zjj}$$

G.4 Supply of components in terms of aggregate exports and prices

Substitute the final equation of G.3 into (7) to obtain

$$EXT_z^* = B_z^X \left\{ \sum_{zj} \beta_{z,zj}^X \left[\left(\frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \right)^{\tau_z^X} EX_{z,zjj} \right]^{\kappa_z^X} \right\}^{\frac{1}{\kappa_z^X}}$$

where $\kappa_z^X = \dfrac{\tau_z^X + 1}{\tau_z^X}$ implies $\tau_z^X = \dfrac{1}{\kappa_z^X - 1}$, and

$$EXT_z^* = B_z^X \left\{ \sum_{zj} \beta_{z,zj}^X \left[\left(\frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \right)^{\frac{1}{\kappa_z^X - 1}} EX_{z,zjj} \right]^{\kappa_z^X} \right\}^{\frac{1}{\kappa_z^X}}$$

$$EXT_z^* = B_z^X EX_{z,zjj} \left\{ \sum_{zj} \beta_{z,zj}^X \left(\frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}} \right)^{\frac{\kappa_z^X}{\kappa_z^X - 1}} \right\}^{\frac{1}{\kappa_z^X}}$$

$$EXT_z^* = B_z^X EX_{z,zjj} \left(\frac{\beta_{z,zjj}^X}{PW_{z,zjj}} \right)^{\frac{1}{\kappa_z^X - 1}} \left\{ \sum_{zj} \beta_{z,zj}^X \left(\frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\frac{\kappa_z^X}{\kappa_z^X - 1}} \right\}^{\frac{1}{\kappa_z^X}}$$

$$\frac{EX_{z,zjj}}{EXT_z^*} = \frac{1}{B_z^X} \left(\frac{PW_{z,zjj}}{\beta_{z,zjj}^X} \right)^{\frac{1}{\kappa_z^X - 1}} \left\{ \sum_{zj} \beta_{z,zj}^X \left(\frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\frac{\kappa_z^X}{\kappa_z^X - 1}} \right\}^{-\frac{1}{\kappa_z^X}}$$

where $\kappa_z^X = \dfrac{\tau_z^X + 1}{\tau_z^X}$ implies $\kappa_z^X - 1 = \dfrac{1}{\tau_z^X}$ and $\dfrac{\kappa_z^X}{\kappa_z^X - 1} = \tau_z^X + 1$. It follows that

$$\frac{EX_{z,zjj}}{EXT_z^*} = \frac{1}{B_z^X}\left(\frac{PW_{z,zjj}}{\beta_{z,zjj}^X}\right)^{\tau_z^X}\left\{\sum_{zj}\beta_{z,zj}^X\left(\frac{PW_{z,zj}}{\beta_{z,zj}^X}\right)^{\tau_z^X+1}\right\}^{-\frac{\tau_z^X}{\tau_z^X+1}}$$

$$\frac{EX_{z,zjj}}{EXT_z^*} = \frac{1}{B_z^X}\left(\frac{PW_{z,zjj}}{\beta_{z,zjj}^X}\right)^{\tau_z^X}\left\{\sum_{zj}PW_{z,zj}\left(\frac{PW_{z,zj}}{\beta_{z,zj}^X}\right)^{\tau_z^X}\right\}^{-\frac{\tau_z^X}{\tau_z^X+1}} \tag{63}$$

where the left-hand side ratio depends only on component prices. As a matter of fact, multiplying both sides of (63) by EXT_z^* transforms it into a supply equation of $EX_{z,zj}$.

G.5 Unit value of aggregate exports

It is clear from (62) that the relative supply on different export markets is independent of the scale of exports, consistent with the first-degree homogeneity of aggregator function (7). Consequently, the unit value of the export aggregate can be obtained from (14) by substituting the optimal ratio determined in (62), and dividing through by EXT_z^*.

$$PXT_z \; EXT_z^* = e_z EX_{z,zjj}\sum_{zj}PW_{z,zj}\left(\frac{EX_{z,zj}}{EX_{z,zjj}}\right)$$

$$PXT_z = e_z\frac{EX_{z,zjj}}{EXT_z^*}\sum_{zj}PW_{z,zj}\left(\frac{\beta_{z,zjj}^X PW_{z,zj}}{\beta_{z,zj}^X PW_{z,zjj}}\right)^{\tau_z^X}$$

$$PXT_z = e_z\frac{EX_{z,zjj}}{EXT_z^*}\left(\frac{\beta_{z,zjj}^X}{PW_{z,zjj}}\right)^{\tau_z^X}\sum_{zj}PW_{z,zj}\left(\frac{PW_{z,zj}}{\beta_{z,zj}^X}\right)^{\tau_z^X}$$

To express PXT_z in terms of the component prices only, we must substitute for $EX_{z,zjj}/EXT_z^*$ in the preceding equation using (63):

$$PXT_z = e_z \frac{1}{B_z^X} \left(\frac{PW_{z,zjj}}{\beta_{z,zjj}^X} \right)^{\tau_z^X} \left\{ \sum_{zj} PW_{z,zj} \left(\frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \right\}^{-\frac{\tau_z^X}{\tau_z^X + 1}}$$

$$\left(\frac{\beta_{z,zjj}^X}{PW_{z,zjj}} \right)^{\tau_z^X} \sum_{zj} PW_{z,zj} \left(\frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X}$$

$$PXT_z = e_z \frac{1}{B_z^X} \left\{ \sum_{zj} PW_{z,zj} \left(\frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \right\}^{1-\frac{\tau_z^X}{\tau_z^X + 1}}$$

$$PXT_z = e_z \frac{1}{B_z^X} \left\{ \sum_{zj} PW_{z,zj} \left(\frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \right\}^{\frac{1}{\tau_z^X + 1}}$$

G.6 Supply of components reformulated

Given the preceding equation, we have

$$B_z^X PXT_z = e_z \left\{ \sum_{zj} PW_{z,zj} \left(\frac{PW_{z,zj}}{\beta_{z,zj}^X} \right)^{\tau_z^X} \right\}^{\frac{1}{\tau_z^X + 1}}$$

So we can rewrite (63) as

$$\frac{EX_{z,zjj}}{EXT_z^*} = \frac{1}{B_z^X} \left(\frac{PW_{z,zjj}}{\beta_{z,zjj}^X} \right)^{\tau_z^X} \left(B_z^X PXT_z \right)^{-\tau_z^X}$$

$$\frac{EX_{z,zjj}}{EXT_z^*} = \left(\frac{1}{B_z^X} \right)^{\tau_z^X + 1} \left(\frac{PW_{z,zjj}}{\beta_{z,zjj}^X PXT_z} \right)^{\tau_z^X}$$

$$EX_{z,zjj} = \left(\frac{1}{B_z^X} \right)^{\tau_z^X + 1} \left(\frac{PW_{z,zjj}}{\beta_{z,zjj}^X PXT_z} \right)^{\tau_z^X} EXT_z^* \qquad (64)$$

which is an alternate form of the supply equation of exports to a particular region.

To obtain equation (34), simply substitute (57) into (64).

APPENDIX H: TWO-REGION VERSION OF MODEL 3

In a 2-region model, each region has only one trading partner, so that, for $zj \neq z$, we have $EXT_z = EX_{z,zj}$, $IMT_z = IM_{zj,z}$, $PW_{zj,z} = PMT_z^*$ and $PW_{z,zj} = PXT_z^*$, which replace (7), (11), (8) and (12) respectively. The first two equalities above, together with (17), imply

$$EXT_{zj} = IMT_z \tag{24}$$

And following two equalities imply

$$PXT_{zj}^* = PMT_z^* \tag{25}$$

So we can insert equations (24) and (25) into the model, and do away with equations (17) and the four equations (7), (11), (8) and (12), and with variables $EX_{z,zj}$, $IM_{z,zj}$ and $PW_{z,zj}$. The $PWINDEX$ variable is consequently re-defined, and equation (20) is replaced by

$$PWINDEX = \sqrt{\frac{\sum_z PMT_z IMT_z^O}{\sum_z PMT_z^O IMT_z^O} \frac{\sum_z PMT_z IMT_z}{\sum_z PMT_z^O IMT_z}} \tag{23}$$

The two-region Model 3 variables are listed in Table H1, and the equations in Table H2. This model has $11N + 1 = 23$ variables and $9N + (N - 1) + 1 = 20$ equations. There are $N + 1 = 3$ degrees of freedom.

Table H1 – Two-region Model 3 variables

Volumes	
Q_z	Domestic demand for the composite good in region *z*
D_z	Domestic demand for the locally produced good in region *z*
IMT_z	Total imports of region *z*
EXT_z	Total exports of region *z*
$CABX_z$	Real current account balance (pseudo-volume variable)
Prices	
P_z^*	Producer price
PL_z^*	Market price of local product
PC_z^*	Price of the composite good
PMT_z^*	Price of composite imports to region *z*
PXT_z^*	Price of composite exports from region *z*
$PW_{z,zj}$	World price of exports from region *z* to region *zj*
Nominal value variables	
CAB_z^*	Current account balance of region *z*

Table H2 – Two-region Model 3 equations

$CAB_z^* = P_z^* XS_z - PC_z^* Q_z,\ z \neq zleon$	(3)
$CABX_z = CAB_z^* / PWINDEX$	(19)
$XS_z = B_z \left[\beta_z D_z^{\kappa_z} + (1 - \beta_z) EXT_z^{\kappa_z} \right]^{\frac{1}{\kappa_z}}$ where $\kappa_z = \dfrac{\tau_z + 1}{\tau_z}$, with $0 < \tau_z < \infty$	(5)
$\dfrac{EXT_z}{D_z} = \left(\dfrac{\beta_z}{1 - \beta_z} \dfrac{PXT_z^*}{PL_z^*} \right)^{\tau_z}$	(6)
$Q_z = A_z \left[\alpha_z D_z^{-\rho_z} + (1 - \alpha_z) IMT_z^{-\rho_z} \right]^{-\frac{1}{\rho_z}}$ where $\rho_z = \dfrac{1 - \sigma_z}{\sigma_z}$, with $0 < \sigma_z < \infty$	(9)
$\dfrac{IMT_z}{D_z} = \left(\dfrac{1 - \alpha_z}{\alpha_z} \dfrac{PL_z}{PMT_z} \right)^{\sigma_z}$	(10)
$P_z^* XS_z = PL_z^* D_z + PXT_z^* EXT_z$	(13)
$PC_z^* Q_z = PL_z^* D_z + PMT_z^* IMT_z$	(15)
$PWINDEX = \sqrt{ \dfrac{\sum_z PMT_z IMT_z^O}{\sum_z PMT_z^O IMT_z^O} \dfrac{\sum_z PMT_z IMT_z}{\sum_z PMT_z^O IMT_z} }$	(23)
$EXT_{zj} = IMT_z$	(24)
$PXT_{zj}^* = PMT_z^*$	(25)

APPENDIX I: EQUATION (26)

The 2-region Q-Model consists of equations (5), (9) and (24). Develop (5) when $XS_z = \overline{XS}_z$

$$\beta_z D_z^{\kappa_z} + (1-\beta_z) IMT_{zj}^{\kappa_z} = \left(\frac{\overline{XS}_z}{B_z}\right)^{\kappa_z}$$

$$D_z^{\kappa_z} = \frac{1}{\beta_z}\left(\frac{\overline{XS}_z}{B_z}\right)^{\kappa_z} - \frac{(1-\beta_z)}{\beta_z} IMT_{zj}^{\kappa_z}$$

$$D_z = \left[\frac{1}{\beta_z}\left(\frac{\overline{XS}_z}{B_z}\right)^{\kappa_z} - \frac{(1-\beta_z)}{\beta_z} IMT_{zj}^{\kappa_z}\right]^{\frac{1}{\kappa_z}} \tag{65}$$

Next, let $Q_z = \overline{Q}_z$, substitute into (9) and develop.

$$\alpha_z D_z^{-\rho_z} + (1-\alpha_z) IMT_z^{-\rho_z} = \left(\frac{\overline{Q}_z}{A_z}\right)^{-\rho_z}$$

$$D_z^{-\rho_z} = \frac{1}{\alpha_z}\left(\frac{\overline{Q}_z}{A_z}\right)^{-\rho_z} - \frac{(1-\alpha_z)}{\alpha_z} IMT_z^{-\rho_z}$$

$$D_z = \left[\frac{1}{\alpha_z}\left(\frac{\overline{Q}_z}{A_z}\right)^{-\rho_z} - \frac{(1-\alpha_z)}{\alpha_z} IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}}$$

Combine that equation with (65)

$$\left[\frac{1}{\alpha_z}\left(\frac{\overline{Q}_z}{A_z}\right)^{-\rho_z} - \frac{(1-\alpha_z)}{\alpha_z} IMT_z^{-\rho_z}\right]^{-\frac{1}{\rho_z}} = \left[\frac{1}{\beta_z}\left(\frac{\overline{XS}_z}{B_z}\right)^{\kappa_z} - \frac{(1-\beta_z)}{\beta_z} IMT_{zj}^{\kappa_z}\right]^{\frac{1}{\kappa_z}}$$

$$\frac{1}{\alpha_z}\left(\frac{\overline{Q}_z}{A_z}\right)^{-\rho_z} - \frac{(1-\alpha_z)}{\alpha_z} IMT_z^{-\rho_z} = \left[\frac{1}{\beta_z}\left(\frac{\overline{XS}_z}{B_z}\right)^{\kappa_z} - \frac{(1-\beta_z)}{\beta_z} IMT_{zj}^{\kappa_z}\right]^{-\frac{\rho_z}{\kappa_z}}$$

$$\frac{(1-\alpha_z)}{\alpha_z} IMT_z^{-\rho_z} = \frac{1}{\alpha_z}\left(\frac{\overline{Q}_z}{A_z}\right)^{-\rho_z} - \left[\frac{1}{\beta_z}\left(\frac{\overline{XS}_z}{B_z}\right)^{\kappa_z} - \frac{(1-\beta_z)}{\beta_z} IMT_{zj}^{\kappa_z}\right]^{-\frac{\rho_z}{\kappa_z}}$$

$$IMT_z^{-\rho_z} = \frac{1}{(1-\alpha_z)}\left(\frac{\overline{Q}_z}{A_z}\right)^{-\rho_z} - \frac{\alpha_z}{(1-\alpha_z)}\left[\frac{1}{\beta_z}\left(\frac{\overline{XS}_z}{B_z}\right)^{\kappa_z} - \frac{(1-\beta_z)}{\beta_z}\,IMT_{zj}^{\kappa_z}\right]^{-\frac{\rho_z}{\kappa_z}}$$

Equation (26) follows directly.

APPENDIX J: EQUATION (27)

The 2-region P-Model consists of equations (6), (10) and (25). Substitute from (25) into (6) and develop.

$$\frac{\beta_z}{1-\beta_z}\frac{PMT^*_{zj}}{PL^*_z} = \left(\frac{EXT_z}{D_z}\right)^{1/\tau_z}$$

$$\frac{1-\beta_z}{\beta_z}\frac{PL^*_z}{PMT^*_{zj}} = \left(\frac{D_z}{EXT_z}\right)^{1/\tau_z}$$

$$PL^*_z = \frac{\beta_z}{1-\beta_z}\left(\frac{D_z}{EXT_z}\right)^{1/\tau_z} PMT^*_{zj} \tag{66}$$

Next, develop (10):

$$\frac{1-\alpha_z}{\alpha_z}\frac{PL^*_z}{PMT^*_z} = \left(\frac{IMT_z}{D_z}\right)^{1/\sigma_z}$$

$$PL^*_z = \frac{\alpha_z}{1-\alpha_z}\left(\frac{IMT_z}{D_z}\right)^{1/\sigma_z} PMT^*_z$$

Combine this with (66):

$$\frac{\alpha_z}{1-\alpha_z}\left(\frac{IMT_z}{D_z}\right)^{1/\sigma_z} PMT^*_z = \frac{\beta_z}{1-\beta_z}\left(\frac{D_z}{EXT_z}\right)^{1/\tau_z} PMT^*_{zj}$$

$$PMT^*_z = \frac{1-\alpha_z}{\alpha_z}\left(\frac{D_z}{IMT_z}\right)^{1/\sigma_z}\frac{\beta_z}{1-\beta_z}\left(\frac{D_z}{EXT_z}\right)^{1/\tau_z} PMT^*_{zj}$$

Equation (27) follows directly.

APPENDIX K: GAMS IMPLEMENTATION

K.1 Brief description of GAMS programs

The GAMS programs can be downloaded from http://www.pep-net.org/training-material-1. They are found under “Other > CGE model closures in a skeleton world model”.

Model 2 has been implemented in GAMS for illustration purposes. The standard version of the model is implemented in program *SkeletonWorld_2014_Model2A.gms*. The main program calls several sub-programs:

- *Calib_check_S2_2014.gms* may be called at the end of calibration. It computes the difference between the left- and right-hand side of every equation when the variable arguments are replaced by their (calibrated) benchmark values.
- *Closures_S2_2014.gms* contains various closure options, including different choices of the numéraire and of its value. The user chooses by activating/disactivating the $ontext/$offtext switches in the program.
- *RESULTS_BAU_S2_2014.gms* stores the BAU (“business as usual”, no shock) solution values.
- *Benchmk_chk_S2_2014.gms* may be called once the BAU solution has been computed. It computes the difference between each variable’s solution value and its benchmark (calibrated) value.
- *RESULTS_SIM_S2_2014.gms* stores the SIM solution values and produces the GDX output file and its xls companion.
- *RATIOS_S2_2014.gms* computes the ratio of SIM solution values to BAU (benchmark) values for the purpose of checking for calibration consistency.

Calib_check_S2_2014.gms and *Benchmk_chk_S2_2014.gms* are not directly related to the issues discussed in this paper. They are two diagnostic tools frequently used by the author in developing models.

The program *SkeletonWorld_2014_Model2A.gms* produces two result files. The first is a “classic” Excel results file created from the standard GDX output file, using GDX2XLS, with one sheet per variable. The second is in tabular form, made with the GDXXRW facility, reading set-up parameters from a text file which is created within the GAMS model program.

There is a second program, *SkeletonWorld_2014_Model2B.gms*, which is basically the same, but offers in addition various possibilities for modifying the calibration and/or the closure rules, and making comparisons with the standard version of the model. Comparisons are made using two sub-programs:

- *Compare_SOLUTIONS_S2.gms* reads the standard model solution GDX file produced by *SkeletonWorld_2014_Model2A.gms*, and computes the ratios of Model2B/Model2A solution values, for the purpose of checking for model homogeneity when the choice of the numeraire, or its value, or both are modified. The output consists of files *Sol_ratios.gdx* and *Sol_ratios.xls*.
- *Compare_RATIOS_S2.gms* reads the GDX file of SIM/BAU ratios produced by *SkeletonWorld_2014_Model2A.gms*, and computes the Model2B–Model2A ratio differences, for the purpose of checking for calibration consistency. The output consists of files *Ratio_diff.gdx* and *Ratio_diff.xls*.

SkeletonWorld_2014_Model2B.gms is therefore a tool for testing model homogeneity and calibration consistency.

K.2 Examples of testso

All the tests described here have been made against the standard version of the model with the FP closure and $PWINDEX$ as the numéraire price:

```
XS.FX(z)      =  XSO(z);
P.FX(z)       =  PO(z);
CABX.FX(z) = CABXO(z);
PWINDEX.FX  = PWINDEXO;
```

In Model2B, the closure is defined in *Closures_S2B_2014.gms*, as described for each test.

K.2.1 Test of FP and FE closures

The FE closure is implemented in Model2B:

```
XS.FX(z)     = XSO(z);
e.FX(z)      = eO(z);
CABX.FX(z)   = CABXO(z);
PWINDEX.FX   = PWINDEXO;
```

In *Compare_SOLUTIONS_S2.gms*, set the Lambda parameter to 1 and the program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical.

K.2.2 Homogeneity test 1

If a model is truly homogenous, the solution values of real (volume) variables and all price and nominal value ratios are supposed to be

- independent of which commodity is taken as the numéraire;
- independent of which region is taken as the reference region when the numéraire is a regional commodity (a particular case of the preceding);
- independent of the particular value given the price of the numéraire, whatever commodity plays that role.

In Model2B, implement the FP closure with $PW_{zr,zrj}$ as the numéraire:

```
zr(z) = no;
zr('Reg1') = yes;
zrj(z) = no;
zrj('Reg2') = yes;

XS.FX(z)      =  XSO(z);
P.FX(z)       =  PO(z);
CABX.FX(z)    =  CABXO(z);
PW.FX(zr,zrj)      =  PWO(zr,zrj);
```

In *Compare_SOLUTIONS_S2.gms*, set the λ parameter using[3]

```
Loop{(zr,zrj),
 Lambda(scen)    = BvalPW(zr,zrj,scen)/valPW(zr,zrj,scen);
     };
```

Then the program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical. As for exchange rates, their ratio is equal to $1/\lambda$. Now, let PL_z^A be the value of PL_z in the Model2A solution, and PL_z^B its value in the Model2B solution. We observe in *Sol_ratios.xls* that $PL_z^B / e_z^B = \lambda\, PL_z^A / e_z^A$ and $e_z^B / e_z^A = 1/\lambda$. Consequently, $PL_z^B / PL_z^A = 1$, which is as it should be under the FP closure. The reason is that the regional numéraires (here P_z) are fixed, so that, if the models are identical, going from $PWINDEX$ to $PW_{zr,zrj}$ as the numéraire leaves regional prices unchanged.

Homogeneity test 1 can be performed with the same results if the numéraire is given any arbitrary positive value. For example,

```
PW.FX(zr,zrj)      =  1.7*PWO(zr,zrj);
```

3 The loop is necessary because *zr* and *zrj* are sets (albeit singletons) in GAMS.

However, with multiples outside the [0.45,1.7] range, the model needs to be initialized accordingly for GAMS to be able to solve it:

```
PW.FX(zr,zrj)      =  Lambda0*PWO(zr,zrj);

e.L(z)        =  eO(z)/Lambda0;
PW.L(z,zj)    =  Lambda0*PWO(z,zj);
PWINDEX.L     =  Lambda0*PWINDEXO;
```

(the re-initialization code appears in *Closures_S2_2014.gms*).

K.2.3 Homogeneity test 2

In Model2B, implement the FP closure with e_{zr} as the numéraire:

```
zr(z) = no;
zr('Reg1') = yes;

XS.FX(z)      =  XSO(z);
P.FX(z)       =  PO(z);
CABX.FX(z)    =  CABXO(z);
e.FX(zr)      =  eO(zr);
```

In *Compare_SOLUTIONS_S2.gms*, set the λ parameter using[4]

```
Loop{zr,
 Lambda(scen)    = vale(zr,scen)/Bvale(zr,scen);
     };
```

which is equivalent to $\lambda = \left(1/e_{zr}^{B}\right)/\left(1/e_{zr}^{A}\right)$. This reflects the fact that the numéraire is an international price: it is actually $1/e_{zr}$, the price of region *zr*'s currency in terms of the international currency, rather than e_{zr}, the price of the international currency in terms of region *zr*'s currency.

The program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical.

Homogeneity test 2 can be performed with the same results if the numéraire is given any arbitrary positive value, but with large or small multiples, the model needs to be initialized accordingly for GAMS to be able to solve it:

---

4 The loop is necessary because *zr* and *zrj* are sets (albeit singletons) in GAMS.

```
e.FX(zr)      =  Lambda0*eO(zr);
e.L(z)        =  Lambda0*eO(z);
PW.L(z,zj)    =  PWO(z,zj)/Lambda0;
PWINDEX.L     =  PWINDEXO/Lambda0;
```

K.2.4 Homogeneity test 3

In Model2B, implement the FP closure with $PWINDEX$ as the numéraire and change P_z for PL_z as the regional numéraire:

```
XS.FX(z)      =  XSO(z);
PL.FX(z)      =  PLO(z);
CABX.FX(z)    =  CABXO(z);
PWINDEX.FX    =  Lambda0*PWINDEXO;
```

and add the corresponding re-initialization:

```
e.L(z)        =  eO(z)/Lambda0;
PW.L(z,zj)    =  Lambda0*PWO(z,zj);
```

In *Compare_SOLUTIONS_S2.gms*, set the λ parameter using

```
Lambda(scen)    = BvalPWINDEX(scen)/valPWINDEX(scen);
```

The program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical.

K.2.5 Homogeneity test 4

In Model2B, implement the FP closure with $PWINDEX$ as the numéraire and change P_z for PL_z as the regional numéraire:

```
XS.FX(z)      =  XSO(z);
PL.FX(z)      =  Lambda0*PLO(z);
CABX.FX(z)    =  CABXO(z);
PWINDEX.FX    =  PWINDEXO;
```

and add the corresponding re-initialization:

```
e.L(z)       =  Lambda0*eO(z);
P.L(z)       =  Lambda0*PO(z);
PC.L(z)      =  Lambda0*PCO(z);
```

```
PMT.L(z)     =  Lambda0*PMTO(z);
PXT.L(z)     =  Lambda0*PXTO(z);
CAB.L(z)     =  Lambda0*CABO(z);
```

In *Compare_SOLUTIONS_S2.gms*, set the λ parameter using

```
Lambda(scen)   = BvalPWINDEX(scen)/valPWINDEX(scen);
```

The program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical.

K.2.6 Homogeneity test 5 with FE closure

The modified FE closure is implemented in Model2B with P_{zr} rather than $PWINDEX$ as numéraire:

```
XS.FX(z)     = XSO(z);
e.FX(z)      = eO(z);
CABX.FX(z)   = CABXO(z);
P.FX(zr)      =  Lambda0*PO(zr);
```

and add the corresponding re-initialization:

```
PL.L(z)       =  Lambda0*PLO(z);
PC.L(z)       =  Lambda0*PCO(z);
PMT.L(z)      =  Lambda0*PMTO(z);
PXT.L(z)      =  Lambda0*PXTO(z);
PW.L(z,zj)    =  Lambda0*PWO(z,zj);
PWINDEX.L     =  Lambda0*PWINDEXO;
CAB.L(z)      =  Lambda0*CABO(z);
```

In *Compare_SOLUTIONS_S2.gms*, set the λ parameter using

```
Loop{zr,
 Lambda(scen) = [BvalP(zr,scen)/Bvale(zr,scen)]
                                        /[valP(zr,scen)/vale(zr,scen)];
     };
```

The program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol_ratios.xls* that all ratios are equal to 1 (except for exchange rates): the two models are identical.

K.2.7 Calibration consistency test of price × volume factoring

Implement the standard FP closure in Model2B with $PWINDEX$ as the numéraire. Then, to perform the test, go to section 4.1 of the *SkeletonWorld_2014_Model2B.gms* main program, look for the line

```
*$GoTo Alternate_calib
```

and remove the asterisk at the beginning. In *Compare_SOLUTIONS_S2.gms*, set the λ parameter using

```
Lambda(scen)    = 1;
```

The program will compute the ratios of Model2B/Model2A solution values, after dividing all regional prices and nominal variables by their exchange rate to convert them into the international currency. Observe in *Sol_ratios.xls* that the ratios of the prices of aggregates (PXT_z, PMT_z, PC_z and P_z) are equal to the Model2B/Model2A ratios of their benchmark values, while the ratios of the corresponding volumes are the inverse, including in the SIM results. In *Ratio_diff.xls*, observe that the SIM/BAU ratios are equal in both models, which satisfies the criterion of calibration consistency as stated in equations (1) and (2) of section 1.3:

K.2.8 Calibration consistency test of arbitrary prices

Implement the standard FP closure in Model2B with $PWINDEX$ as the numéraire.

In section 3.4 of the *SkeletonWorld_2014_Model2B.gms* program, replace the following statements

```
eO(z)        = 1;
PLO(z)       = 1;
PWO(z,zj)    = 1;
```

with different assignments. For example:

```
eO(z)        = .5;
PLO(z)       = 0.8;
PWO(z,zj)    = 1.5;
PLO('Reg3')= 1.6;
PWO(zr,zrj)= 2;
```

In *Sol_ratios.xls*, the SIM ratios and the BAU ratios are equal, and the passionate reader could trace the sources of divergence from 1. More interesting is *Ratio_diff.xls*, where it can be verified that the SIM/BAU ratios are equal in both models, which satisfies the criterion of calibration consistency as stated in section 1.3.

K.2.9 Calibration consistency test of arbitrary prices and exchange rates

Implement the standard FP closure in Model2B with $PWINDEX$ as the numéraire. In the *SkeletonWorld_2014_Model2B.gms* program, go to the section labeled "4.1_supplement Re-calibration with arbitrary unequal exchange rates", and activate the procedure by cancelling the $ontext/$offtext switches. The same result obtains as in the preceding test.